

9

Transformations

What You'll Learn

- **Lesson 9-1, 9-2, 9-3, and 9-5** Name, draw, and recognize figures that have been reflected, translated, rotated, or dilated.
- **Lesson 9-4** Identify and create different types of tessellations.
- **Lesson 9-6** Find the magnitude and direction of vectors and perform operations on vectors.
- **Lesson 9-7** Use matrices to perform transformations on the coordinate plane.

Key Vocabulary

- reflection (p. 463)
- translation (p. 470)
- rotation (p. 476)
- tessellation (p. 483)
- dilation (p. 490)
- vector (p. 498)

Why It's Important

Transformations, lines of symmetry, and tessellations can be seen in artwork, nature, interior design, quilts, amusement parks, and marching band performances. These geometric procedures and characteristics make objects more visually pleasing. *You will learn how mosaics are created by using transformations in Lesson 9-2.*

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 9.

For Lessons 9-1 through 9-5

Graph Points

Graph each pair of points. (For review, see pages 728 and 729.)

- $A(1, 3), B(-1, 3)$
- $C(-3, 2), D(-3, -2)$
- $E(-2, 1), F(-1, -2)$
- $G(2, 5), H(5, -2)$
- $J(-7, 10), K(-6, 7)$
- $L(3, -2), M(6, -4)$

For Lesson 9-6

Distance and Slope

Find $m\angle A$. Round to the nearest tenth. (For review, see Lesson 7-4.)

- $\tan A = \frac{3}{4}$
- $\tan A = \frac{5}{8}$
- $\sin A = \frac{2}{3}$
- $\sin A = \frac{4}{5}$
- $\cos A = \frac{9}{12}$
- $\cos A = \frac{15}{17}$

For Lesson 9-7

Multiply Matrices

Find each product. (For review, see pages 752 and 753.)

- $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 4 \\ -5 & -1 \end{bmatrix}$
- $\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -2 \\ -2 & 3 \end{bmatrix}$
- $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 & 4 & 5 \\ -2 & -5 & 1 \end{bmatrix}$
- $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -3 & -3 & 2 \\ 3 & -1 & -2 & 1 \end{bmatrix}$

FOLDABLES™ Study Organizer

Transformations Make this Foldable to help you organize the types of transformations. Begin with one sheet of notebook paper.

Step 1 Fold

Fold a sheet of notebook paper in half lengthwise.



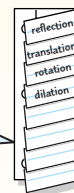
Step 2 Cut

Cut the flap on every third line.



Step 3 Label

Label each tab with a vocabulary word from this chapter.



Reading and Writing As you read and study the chapter, use each page to write notes and examples of transformations, tessellations, and vectors on the coordinate plane.



Geometry Activity

A Preview of Lesson 9-1

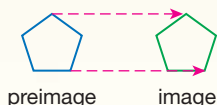
Transformations

In a plane, you can slide, flip, turn, enlarge, or reduce figures to create new figures. These corresponding figures are frequently designed into wallpaper borders, mosaics, and artwork. Each figure that you see will correspond to another figure. These corresponding figures are formed using transformations.

A **transformation** maps an initial image, called a preimage, onto a final image, called an image. Below are some of the types of transformations. The red lines show some corresponding points.

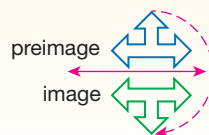
translation

A figure can be slid in any direction.



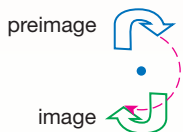
reflection

A figure can be flipped over a line.



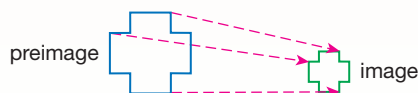
rotation

A figure can be turned around a point.

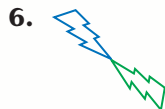
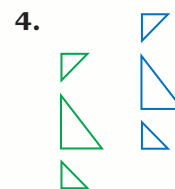


dilation

A figure can be enlarged or reduced.



Exercises Identify the following transformations. The blue figure is the preimage.



Make a Conjecture

11. An *isometry* is a transformation in which the resulting image is congruent to the preimage. Which transformations are isometries?



9-1 Reflections

What You'll Learn

- Draw reflected images.
- Recognize and draw lines of symmetry and points of symmetry.

Vocabulary

- reflection
- line of reflection
- isometry
- line of symmetry
- point of symmetry

Where are reflections found in nature?

On a clear, bright day glacial-fed lakes can provide vivid reflections of the surrounding vistas. Note that each point above the water line has a corresponding point in the image in the lake. The distance that a point lies above the water line appears the same as the distance its image lies below the water.



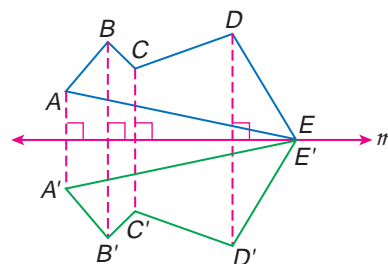
Study Tip

Reading Math

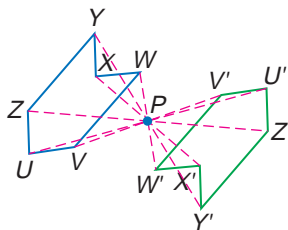
A' , A'' , A''' , and so on name corresponding points for one or more transformations.

DRAW REFLECTIONS A **reflection** is a transformation representing a flip of a figure. Figures may be reflected in a point, a line, or a plane.

The figure shows a reflection of $ABCDE$ in line m . Note that the segment connecting a point and its image is perpendicular to line m and is bisected by line m . Line m is called the **line of reflection** for $ABCDE$ and its image $A'B'C'D'E'$. Because E lies on the line of reflection, its preimage and image are the same point.



It is possible to reflect a preimage in a point. In the figure below, polygon $UVWXYZ$ is reflected in point P .



Note that P is the midpoint of each segment connecting a point with its image.

$$\begin{aligned} \overline{UP} &\cong \overline{PU'}, \overline{VP} \cong \overline{PV'}, \\ \overline{WP} &\cong \overline{PW'}, \overline{XP} \cong \overline{PX'}, \\ \overline{YP} &\cong \overline{PY'}, \overline{ZP} \cong \overline{PZ'} \end{aligned}$$

Study Tip

Look Back

To review **congruence transformations**, see Lesson 4-3.

When reflecting a figure in a line or in a point, the image is congruent to the preimage. Thus, a reflection is a **congruence transformation**, or an **isometry**. That is, reflections preserve distance, angle measure, betweenness of points, and collinearity. In the figure above, polygon $UVWXYZ \cong$ polygon $U'V'W'X'Y'Z'$.

Corresponding Sides	Corresponding Angles
$\overline{UV} \cong \overline{U'V'}$	$\angle UVW \cong \angle U'V'W'$
$\overline{VW} \cong \overline{V'W'}$	$\angle VWX \cong \angle V'W'X'$
$\overline{WX} \cong \overline{W'X'}$	$\angle WXY \cong \angle W'X'Y'$
$\overline{XY} \cong \overline{X'Y'}$	$\angle XYZ \cong \angle X'Y'Z'$
$\overline{YZ} \cong \overline{Y'Z'}$	$\angle YZU \cong \angle Y'Z'U'$
$\overline{UZ} \cong \overline{U'Z'}$	$\angle ZUV \cong \angle Z'U'V'$

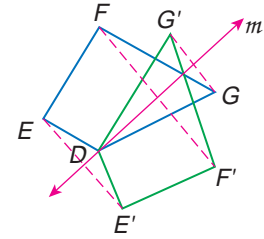
Example 1 Reflecting a Figure in a Line

Draw the reflected image of quadrilateral $DEFG$ in line m .

Step 1 Since D is on line m , D is its own reflection. Draw segments perpendicular to line m from E , F , and G .

Step 2 Locate E' , F' , and G' so that line m is the perpendicular bisector of $\overline{EE'}$, $\overline{FF'}$, and $\overline{GG'}$. Points E' , F' , and G' are the respective images of E , F , and G .

Step 3 Connect vertices D , E' , F' , and G' .



Since points D , E' , F' , and G' are the images of points D , E , F , and G under reflection in line m , then quadrilateral $DE'F'G'$ is the reflection of quadrilateral $DEFG$ in line m .

Reflections can also occur in the coordinate plane.

Example 2 Reflection in the x -axis

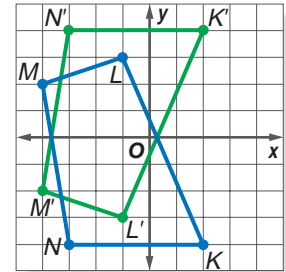
COORDINATE GEOMETRY Quadrilateral $KLMN$ has vertices $K(2, -4)$, $L(-1, 3)$, $M(-4, 2)$, and $N(-3, -4)$. Graph $KLMN$ and its image under reflection in the x -axis. Compare the coordinates of each vertex with the coordinates of its image.

Use the vertical grid lines to find a corresponding point for each vertex so that the x -axis is equidistant from each vertex and its image.

$$K(2, -4) \rightarrow K'(2, 4) \qquad L(-1, 3) \rightarrow L'(-1, -3)$$

$$M(-4, 2) \rightarrow M'(-4, -2) \qquad N(-3, -4) \rightarrow N'(-3, 4)$$

Plot the reflected vertices and connect to form the image $K'L'M'N'$. The x -coordinates stay the same, but the y -coordinates are opposite. That is, $(a, b) \rightarrow (a, -b)$.



Example 3 Reflection in the y -axis

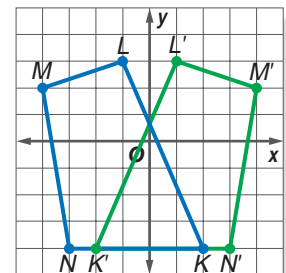
COORDINATE GEOMETRY Suppose quadrilateral $KLMN$ from Example 2 is reflected in the y -axis. Graph $KLMN$ and its image under reflection in the y -axis. Compare the coordinates of each vertex with the coordinates of its image.

Use the horizontal grid lines to find a corresponding point for each vertex so that the y -axis is equidistant from each vertex and its image.

$$K(2, -4) \rightarrow K'(-2, -4) \qquad L(-1, 3) \rightarrow L'(1, 3)$$

$$M(-4, 2) \rightarrow M'(4, 2) \qquad N(-3, -4) \rightarrow N'(3, -4)$$

Plot the reflected vertices and connect to form the image $K'L'M'N'$. The x -coordinates are opposite and the y -coordinates are the same. That is, $(a, b) \rightarrow (-a, b)$.



Study Tip

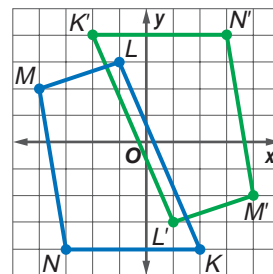
Reading Mathematics

The expression $K(2, -4) \rightarrow K'(2, 4)$ can be read as “point K is mapped to new location K' .” This means that point K' in the image corresponds to point K in the preimage.

Example 4 Reflection in the Origin

COORDINATE GEOMETRY Suppose quadrilateral $KLMN$ from Example 2 is reflected in the origin. Graph $KLMN$ and its image under reflection in the origin. Compare the coordinates of each vertex with the coordinates of its image.

Since $\overline{KK'}$ passes through the origin, use the horizontal and vertical distances from K to the origin to find the coordinates of K' . From K to the origin is 4 units up and 2 units left. K' is located by repeating that pattern from the origin. Four units up and 2 units left yields $K'(-2, 4)$.



$$K(2, -4) \rightarrow K'(-2, 4) \quad L(-1, 3) \rightarrow L'(1, -3)$$

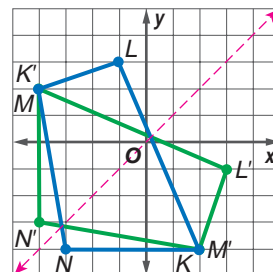
$$M(-4, 2) \rightarrow M'(4, -2) \quad N(-3, -4) \rightarrow N'(3, 4)$$

Plot the reflected vertices and connect to form the image $K'L'M'N'$. Comparing coordinates shows that $(a, b) \rightarrow (-a, -b)$.

Example 5 Reflection in the Line $y = x$

COORDINATE GEOMETRY Suppose quadrilateral $KLMN$ from Example 2 is reflected in the line $y = x$. Graph $KLMN$ and its image under reflection in the line $y = x$. Compare the coordinates of each vertex with the coordinates of its image.

The slope of $y = x$ is 1. $\overline{KK'}$ is perpendicular to $y = x$, so its slope is -1 . From K to the line $y = x$, move up three units and left three units. From the line $y = x$ move up three units and left three units to $K'(-4, 2)$.



$$K(2, -4) \rightarrow K'(-4, 2) \quad L(-1, 3) \rightarrow L'(3, -1)$$

$$M(-4, 2) \rightarrow M'(2, -4) \quad N(-3, -4) \rightarrow N'(-4, -3)$$

Plot the reflected vertices and connect to form the image $K'L'M'N'$. Comparing coordinates shows that $(a, b) \rightarrow (b, a)$.

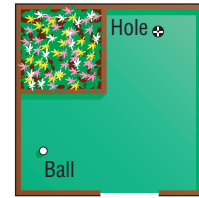
Concept Summary

Reflections in the Coordinate Plane

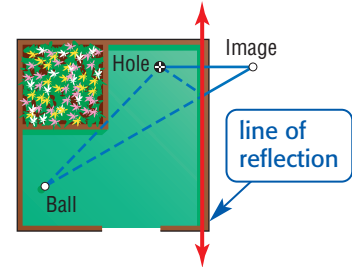
Reflection	x -axis	y -axis	origin	$y = x$
Preimage to Image	$(a, b) \rightarrow (a, -b)$	$(a, b) \rightarrow (-a, b)$	$(a, b) \rightarrow (-a, -b)$	$(a, b) \rightarrow (b, a)$
How to find coordinates	Multiply the y -coordinate by -1 .	Multiply the x -coordinate by -1 .	Multiply both coordinates by -1 .	Interchange the x - and y -coordinates.
Example				

Example 6 Use Reflections

GOLF Adeel and Natalie are playing miniature golf. Adeel says that he read how to use reflections to help make a hole-in-one on most miniature golf holes. Describe how he should putt the ball to make a hole-in-one.



If Adeel tries to putt the ball directly to the hole, he will strike the border as indicated by the blue line. So, he can mentally reflect the hole in the line that contains the right border. If he putts the ball at the reflected image of the hole, the ball will strike the border, and it will rebound on a path toward the hole.



Study Tip

A Point of Symmetry

A point of symmetry is the intersection of all the segments joining a preimage to an image.

LINES AND POINTS OF SYMMETRY Some figures can be folded so that the two halves match exactly. The fold is a line of reflection called a **line of symmetry**. For some figures, a point can be found that is a common point of reflection for all points on a figure. This common point of reflection is called a **point of symmetry**.

Lines of Symmetry

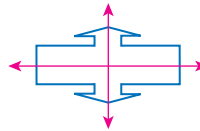
None

F

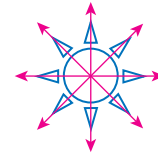
One



Two



More than Two



Points of Symmetry

F

No points of symmetry.



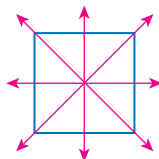
Each point on the figure must have an image on the figure for a point of symmetry to exist.



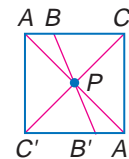
A point of symmetry is the midpoint of all segments between the preimage and the image.

Example 7 Draw Lines of Symmetry

Determine how many lines of symmetry a square has. Then determine whether a square has point symmetry.



A square has four lines of symmetry.



A square has point symmetry. P is the point of symmetry such that $AP = PA'$, $BP = PB'$, $CP = PC'$, and so on.

Check for Understanding

Concept Check

1. Find a counterexample to disprove the statement *A point of symmetry is the intersection of two or more lines of symmetry for a plane figure.*
2. **OPEN ENDED** Draw a figure on the coordinate plane and then reflect it in the line $y = x$. Label the coordinates of the preimage and the image.
3. Identify four properties that are preserved in reflections.

Guided Practice

4. Copy the figure at the right. Draw its reflected image in line m .

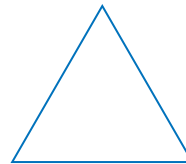


Determine how many lines of symmetry each figure has. Then determine whether the figure has point symmetry.

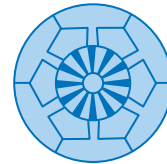
5.



6.



7.



COORDINATE GEOMETRY Graph each figure and its image under the given reflection.

8. \overline{AB} with endpoints $A(2, 4)$ and $B(-3, -3)$ reflected in the x -axis
9. $\triangle ABC$ with vertices $A(-1, 4)$, $B(4, -2)$, and $C(0, -3)$ reflected in the y -axis
10. $\triangle DEF$ with vertices $D(-1, -3)$, $E(3, -2)$, and $F(1, 1)$ reflected in the origin
11. $\square GHIJ$ with vertices $G(-1, 2)$, $H(2, 3)$, $I(6, 1)$, and $J(3, 0)$ reflected in the line $y = x$

Application

NATURE Determine how many lines of symmetry each object has. Then determine whether each object has point symmetry.

12.



13.



14.



Practice and Apply

Homework Help

For Exercises	See Examples
15–26, 38, 39	1
29, 34, 40, 41	2
30, 33, 41	3
27, 28	4
31, 32, 40	5
42	6
35–37, 44–47	7

Extra Practice
See page 771.

Refer to the figure at the right. Name the image of each figure under a reflection in:

line ℓ

15. \overline{WX}

16. \overline{WZ}

17. $\angle XZY$

line m

18. T

19. \overline{UY}

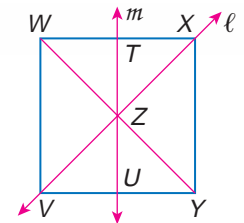
20. $\triangle YVW$

point Z

21. U

22. $\angle TXZ$

23. $\triangle YUZ$

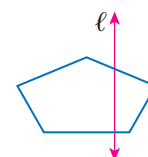


Copy each figure. Draw the image of each figure under a reflection in line ℓ .

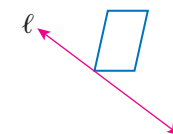
24.



25.



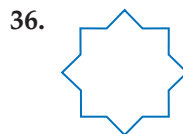
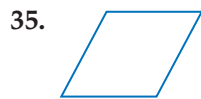
26.



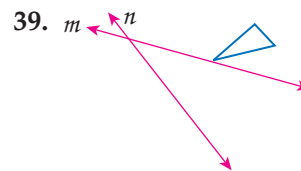
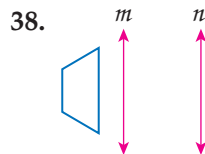
COORDINATE GEOMETRY Graph each figure and its image under the given reflection.

27. rectangle $MNPQ$ with vertices $M(2, 3)$, $N(2, -3)$, $P(-2, -3)$, and $Q(-2, 3)$ in the origin
28. quadrilateral $GHIJ$ with vertices $G(-2, -2)$, $H(2, 0)$, $I(3, 3)$, and $J(-2, 4)$ in the origin
29. square $QRST$ with vertices $Q(-1, 4)$, $R(2, 5)$, $S(3, 2)$, and $T(0, 1)$ in the x -axis
30. trapezoid with vertices $D(4, 0)$, $E(-2, 4)$, $F(-2, -1)$, and $G(4, -3)$ in the y -axis
31. $\triangle BCD$ with vertices $B(5, 0)$, $C(-2, 4)$, and $D(-2, -1)$ in the line $y = x$
32. $\triangle KLM$ with vertices $K(4, 0)$, $L(-2, 4)$, and $M(-2, 1)$ in the line $y = 2$
33. The reflected image of $\triangle FGH$ has vertices $F'(1, 4)$, $G'(4, 2)$, and $H'(3, -2)$. Describe the reflection in the y -axis.
34. The reflected image of $\triangle XYZ$ has vertices $X'(1, 4)$, $Y'(2, 2)$, and $Z'(-2, -3)$. Describe the reflection in the line $x = -1$.

Determine how many lines of symmetry each figure has. Then determine whether the figure has point symmetry.



Copy each figure and then reflect the figure in line m first and then reflect that image in line n . Compare the preimage with the final image.



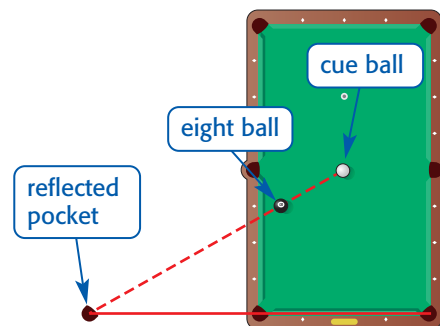
40. **COORDINATE GEOMETRY** Square $DEFG$ with vertices $D(-1, 4)$, $E(2, 8)$, $F(6, 5)$, and $G(3, 1)$ is reflected first in the x -axis, then in the line $y = x$. Find the coordinates of $D''E''F''G''$.
41. **COORDINATE GEOMETRY** Triangle ABC has been reflected in the x -axis, then the y -axis, then the origin. The result has coordinates $A'''(4, 7)$, $B'''(10, -3)$, and $C'''(-6, -8)$. Find the coordinates of A , B , and C .

Billiards

The game in its present form was popular in the early 1800s, but games similar to billiards appeared as early as the 14th century. There are three types of billiards: carom billiards, pocket billiards (pool), and snooker.

Source: www.infoplease.com

42. **BILLIARDS** Tonya is playing billiards. She wants to pocket the eight ball in the lower right pocket using the white cue ball. Copy the diagram and sketch the path the eight ball must travel after being struck by the cue ball.



43. **CRITICAL THINKING** Show that the image of a point upon reflection in the origin is the same image obtained when reflecting a point in the x -axis and then the y -axis.

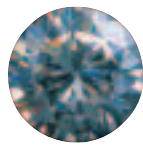
WebQuest

A reflection of the quadrilateral on the map will help you come closer to locating the hidden treasure. Visit www.geometryonline.com/webquest to continue work on your WebQuest project.

DIAMONDS For Exercises 44–47, use the following information.

Diamond jewelers offer a variety of cuts. For each top view, identify any lines or points of symmetry.

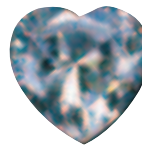
44. round cut



45. pear cut



46. heart cut



47. emerald cut



48. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

Where are reflections found in nature?

Include the following in your answer:

- three examples in nature having line symmetry, and
- an explanation of how the distance from each point above the water line relates to the image in the water.

Standardized Test Practice

A B C D

49. The image of $A(-2, 5)$ under a reflection is $A'(2, -5)$. Which reflection or group of reflections was used?

I. reflected in the x -axis II. reflected in the y -axis III. reflected in the origin

(A) I or III (B) II and III (C) I and II (D) I and II or III

50. **ALGEBRA** If $a \star c = 2a + b + 2c$, find $a \star c$ when $a = 25$, $b = 18$, and $c = 45$.

(A) 176 (B) 158 (C) 133 (D) 88

Maintain Your Skills

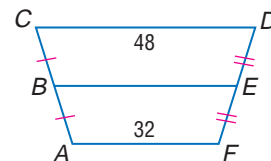
Mixed Review

Write a coordinate proof for each of the following. (Lesson 8-7)

51. The segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.
52. The segments joining the midpoints of the sides of an isosceles trapezoid form a rhombus.

Refer to trapezoid $ACDF$. (Lesson 8-6)

53. Find BE .
54. Let \overline{XY} be the median of $BCDE$. Find XY .
55. Let \overline{WZ} be the median of $ABEF$. Find WZ .



Solve each $\triangle FGH$ described below. Round angle measures to the nearest degree and side measures to the nearest tenth. (Lesson 7-6)

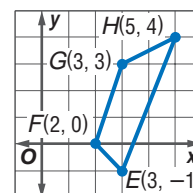
56. $m\angle G = 53$, $m\angle H = 71$, $f = 48$
57. $g = 21$, $m\angle G = 45$, $m\angle F = 59$
58. $h = 13.2$, $m\angle H = 61$, $f = 14.5$

Getting Ready for the Next Lesson

PREREQUISITE SKILL Find the exact length of each side of quadrilateral $EFGH$.

(To review the Distance Formula, see Lesson 3-3.)

59. \overline{EF} 60. \overline{FG}
61. \overline{GH} 62. \overline{HE}



9-2 Translations

What You'll Learn

- Draw translated images using coordinates.
- Draw translated images by using repeated reflections.

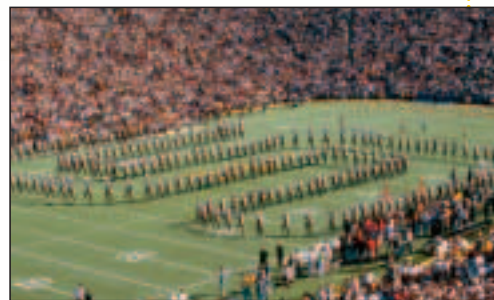
Vocabulary

- translation
- composition of reflections
- glide reflection

How are translations used in a marching band show?

The sights and pageantry of a marching band performance can add to the excitement of a sporting event. Band members dedicate a lot of time and energy to learning the music as well as the movements required for a performance.

The movements of each band member as they progress through the show are examples of *translations*.



Study Tip

Reading Math

A translation is also called a slide, a shift, or a glide.

TRANSLATIONS USING COORDINATES A **translation** is a transformation that moves all points of a figure the same distance in the same direction. Translations on the coordinate plane can be drawn if you know the direction and how far the figure is moving horizontally and/or vertically. For the fixed values of a and b , a translation moves every point $P(x, y)$ of a plane figure to an image $P'(x + a, y + b)$. One way to symbolize a transformation is to write $(x, y) \rightarrow (x + a, y + b)$.

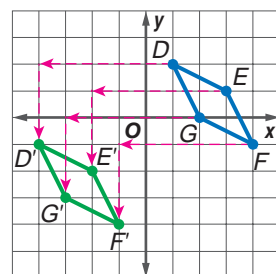
In the figure, quadrilateral $DEFG$ has been translated 5 units to the left and three units down. This can be written as $(x, y) \rightarrow (x - 5, y - 3)$.

$$D(1, 2) \rightarrow D'(1 - 5, 2 - 3) \text{ or } D'(-4, -1)$$

$$E(3, 1) \rightarrow E'(3 - 5, 1 - 3) \text{ or } E'(-2, -2)$$

$$F(4, -1) \rightarrow F'(4 - 5, -1 - 3) \text{ or } F'(-1, -4)$$

$$G(2, 0) \rightarrow G'(2 - 5, 0 - 3) \text{ or } G'(-3, -3)$$



Example 1 Translations in the Coordinate Plane

Rectangle $PQRS$ has vertices $P(-3, 5)$, $Q(-4, 2)$, $R(3, 0)$, and $S(4, 3)$. Graph $PQRS$ and its image for the translation $(x, y) \rightarrow (x + 8, y - 5)$.

This translation moved every point of the preimage 8 units right and 5 units down.

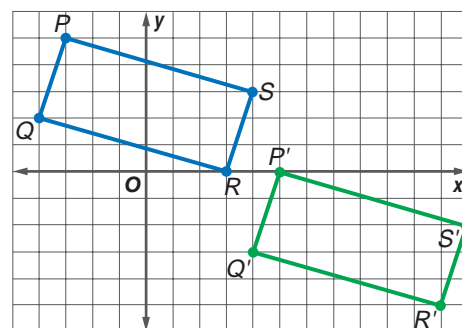
$$P(-3, 5) \rightarrow P'(-3 + 8, 5 - 5) \text{ or } P'(5, 0)$$

$$Q(-4, 2) \rightarrow Q'(-4 + 8, 2 - 5) \text{ or } Q'(4, -3)$$

$$R(3, 0) \rightarrow R'(3 + 8, 0 - 5) \text{ or } R'(11, -5)$$

$$S(4, 3) \rightarrow S'(4 + 8, 3 - 5) \text{ or } S'(12, -2)$$

Plot the translated vertices and connect to form rectangle $P'Q'R'S'$.



Study Tip

Counting Method

Each image point can be located using the same procedure as finding points on a line using the slope.

More About . . .

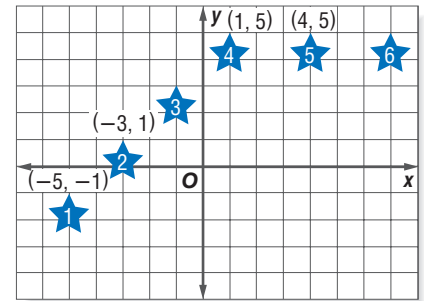


Animation

Animation began as a series of hand-drawn cells, each one with a subtle change made in the drawing. When the drawings were displayed in rapid succession, it visually created movement.

Example 2 Repeated Translations

ANIMATION Computers are often used to create animation. The graph shows repeated translations that result in animation of the star. Find the translation that moves star 1 to star 2 and the translation that moves star 4 to star 5.



To find the translation from star 1 to star 2, use the coordinates at the top of each star. Use the coordinates $(-5, -1)$ and $(-3, 1)$ in the formula.

$$(x, y) \rightarrow (x + a, y + b)$$

$$(-5, -1) \rightarrow (-3, 1)$$

$$\begin{aligned} x + a &= -3 \\ -5 + a &= -3 & x &= -5 \\ a &= 2 & \text{Add 5 to each side.} \end{aligned}$$

$$\begin{aligned} y + b &= 1 \\ -1 + b &= 1 & y &= -1 \\ b &= 2 & \text{Add 1 to each side.} \end{aligned}$$

The translation is $(x + 2, y + 2)$.

Use the coordinates $(1, 5)$ and $(4, 5)$ to find the translation from star 4 to star 5.

$$(x, y) \rightarrow (x + a, y + b)$$

$$(1, 5) \rightarrow (4, 5)$$

$$\begin{aligned} x + a &= 4 \\ 1 + a &= 4 & x &= 1 \\ a &= 3 & \text{Subtract 1 from each side.} \end{aligned}$$

$$\begin{aligned} y + b &= 5 \\ 5 + b &= 5 & y &= 5 \\ b &= 0 & \text{Subtract 5 from each side.} \end{aligned}$$

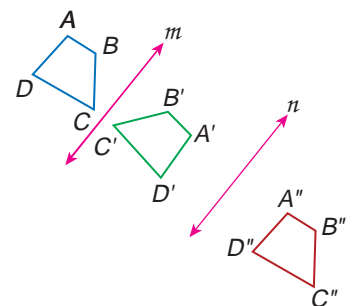
The translation is $(x + 3, y)$ from star 4 to star 5 and from star 5 to star 6.

TRANSLATIONS BY REPEATED REFLECTIONS Another way to find a translation is to perform a reflection in the first of two parallel lines and then reflect the image in the other parallel line. Successive reflections in parallel lines are called a **composition of reflections**.

Example 3 Find a Translation Using Reflections

In the figure, lines m and n are parallel. Determine whether the red figure is a translation image of the blue preimage, quadrilateral $ABCD$.

Reflect quadrilateral $ABCD$ in line m . The result is the green image, quadrilateral $A'B'C'D'$. Then reflect the green image, quadrilateral $A'B'C'D'$ in line n . The red image, quadrilateral $A''B''C''D''$, has the same orientation as quadrilateral $ABCD$.



Quadrilateral $A''B''C''D''$ is the translation image of quadrilateral $ABCD$.

Since translations are compositions of two reflections, all translations are isometries. Thus, all properties preserved by reflections are preserved by translations. These properties include betweenness of points, collinearity, and angle and distance measure.

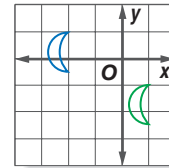


Check for Understanding

- Concept Check**
- OPEN ENDED** Choose integer coordinates for any two points A and B . Then describe how you could count to find the translation of point A to point B .
 - Explain** which properties are preserved in a translation and why they are preserved.
 - FIND THE ERROR** Allie and Tyrone are describing the transformation in the drawing.

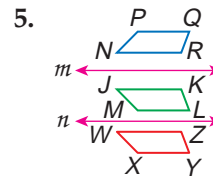
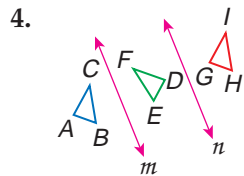
Allie
This is a translation right 3 units and down 2 units.

Tyrone
This is a reflection in the y -axis and then the x -axis.



Who is correct? Explain your reasoning.

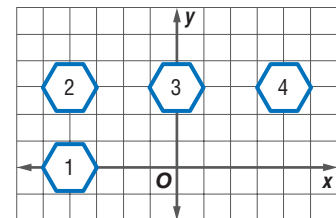
Guided Practice In each figure, $m \parallel n$. Determine whether the red figure is a translation image of the blue figure. Write *yes* or *no*. Explain your answer.



Application **COORDINATE GEOMETRY** Graph each figure and its image under the given translation.

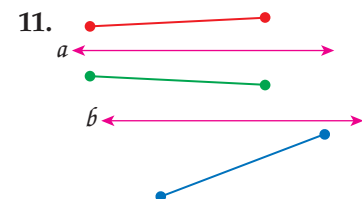
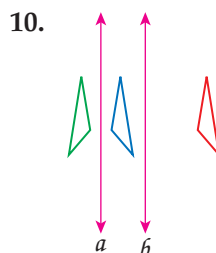
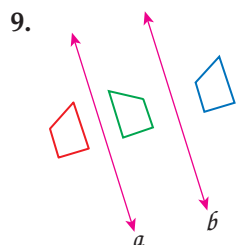
- \overline{DE} with endpoints $D(-3, -4)$ and $E(4, 2)$ under the translation $(x, y) \rightarrow (x + 1, y + 3)$
- $\triangle KLM$ with vertices $K(5, -2)$, $L(-3, -1)$, and $M(0, 5)$ under the translation $(x, y) \rightarrow (x - 3, y - 4)$

8. **ANIMATION** Find the translations that move the hexagon on the coordinate plane in the order given.



Practice and Apply

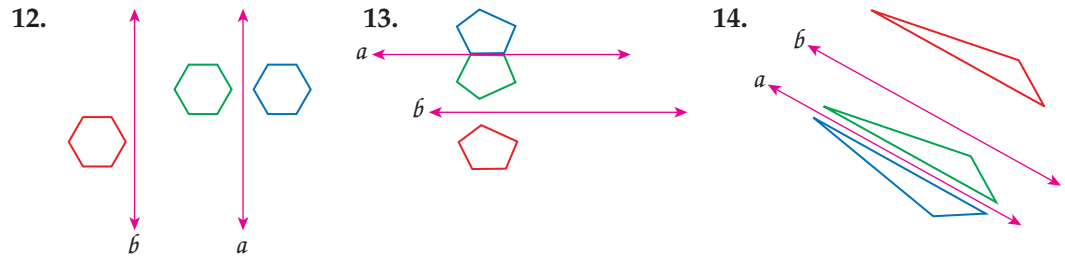
In each figure, $a \parallel b$. Determine whether the red figure is a translation image of the blue figure. Write *yes* or *no*. Explain your answer.



Homework Help

For Exercises	See Examples
9–14, 28, 29	3
15–20, 24–27	1
19, 21–23	2

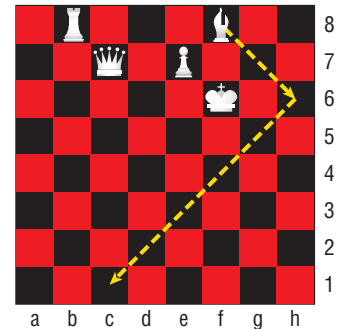
Extra Practice
See page 771.



COORDINATE GEOMETRY Graph each figure and its image under the given translation.

- \overline{PQ} with endpoints $P(2, -4)$ and $Q(4, 2)$ under the translation left 3 units and up 4 units
- \overline{AB} with endpoints $A(-3, 7)$ and $B(-6, -6)$ under the translation 4 units to the right and down 2 units
- $\triangle MJP$ with vertices $M(-2, -2)$, $J(-5, 2)$, and $P(0, 4)$ under the translation $(x, y) \rightarrow (x + 1, y + 4)$
- $\triangle EFG$ with vertices $E(0, -4)$, $F(-4, -4)$, and $G(0, 2)$ under the translation $(x, y) \rightarrow (x + 2, y - 1)$
- quadrilateral $PQRS$ with vertices $P(1, 4)$, $Q(-1, 4)$, $R(-2, -4)$, and $S(2, -4)$ under the translation $(x, y) \rightarrow (x - 5, y + 3)$
- pentagon $VWXYZ$ with vertices $V(-3, 0)$, $W(-3, 2)$, $X(-2, 3)$, $Y(0, 2)$, and $Z(-1, 0)$ under the translation $(x, y) \rightarrow (x + 4, y - 3)$

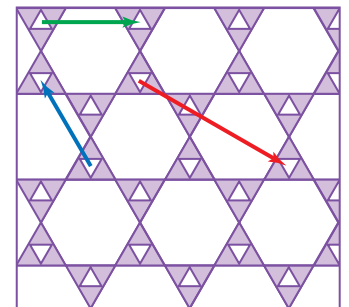
- CHESSE** The bishop shown in square f8 can only move diagonally along dark squares. If the bishop is in c1 after two moves, describe the translation.
- RESEARCH** Use the Internet or other resource to write a possible translation for each chess piece for a single move.



MOSAICS For Exercises 23–25, use the following information.

The mosaic tiling shown on the right is a thirteenth-century Roman inlaid marble tiling. Suppose this pattern is a floor design where the length of the small white equilateral triangle is 12 inches. All triangles and hexagons are regular. Describe the translations in inches represented by each line.

- green line
- blue line
- red line



- CRITICAL THINKING** Triangle TWY has vertices $T(3, -7)$, $W(7, -4)$, and $Y(9, -8)$. Triangle BDG has vertices $B(3, 3)$, $D(7, 6)$, and $G(9, 2)$. If $\triangle BDG$ is the translation image of $\triangle TWY$ with respect to two parallel lines, find the equations that represent two possible parallel lines.

Study Tip

Coordinate Translations

To check your graph, trace the original image and cut it out. Place it on the original graph and, keeping the same orientation, move the copy the correct number of spaces up/down and right/left. Does the copy match your image?

COORDINATE GEOMETRY Graph each figure and the image under the given translation.

- $\triangle PQR$ with vertices $P(-3, -2)$, $Q(-1, 4)$, and $R(2, -2)$ under the translation $(x, y) \rightarrow (x + 2, y - 4)$
- $\triangle RST$ with vertices $R(-4, -1)$, $S(-1, 3)$, and $T(-1, 1)$ reflected in $y = 2$ and then reflected in $y = -2$
- Under $(x, y) \rightarrow (x - 4, y + 5)$, $\triangle ABC$ has translated vertices $A'(-8, 5)$, $B'(2, 7)$, and $C'(3, 1)$. Find the coordinates of A , B , and C .
- Triangle FGH is translated to $\triangle MNP$. Given $F(3, 9)$, $G(-1, 4)$, $M(4, 2)$, and $P(6, -3)$, find the coordinates of H and N . Then write the coordinate form of the translation.

STUDENTS For Exercises 31–33, refer to the graphic at the right. Each bar of the graph is made up of a boy-girl-boy unit.

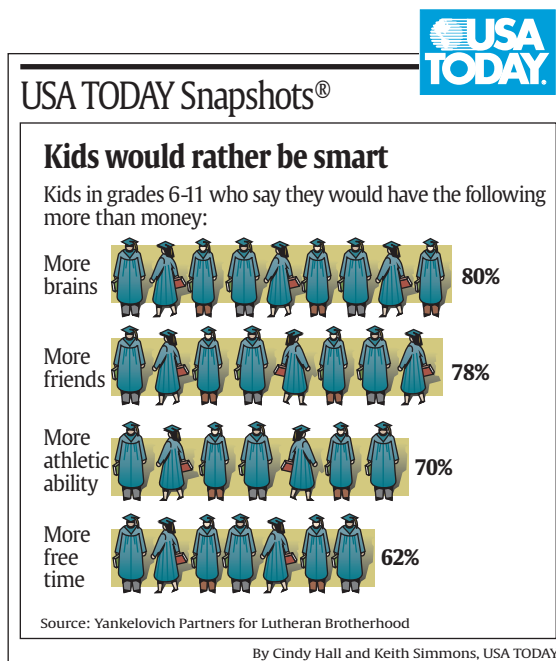
- Which categories show a boy-girl-boy unit that is translated within the bar?
- Which categories show a boy-girl-boy unit that is reflected within the bar?
- Does each person shown represent the same percent? Explain.



Online Research

Data Update How much allowance do teens receive?

Visit www.geometryonline.com/data_update to learn more.



- WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are translations used in a marching band show?

Include the following in your answer:

- the types of movements by band members that are translations, and
- a sketch of a simple pattern for a band member.

Extending the Lesson

GLIDE REFLECTION For Exercises 35–37, use the following information.

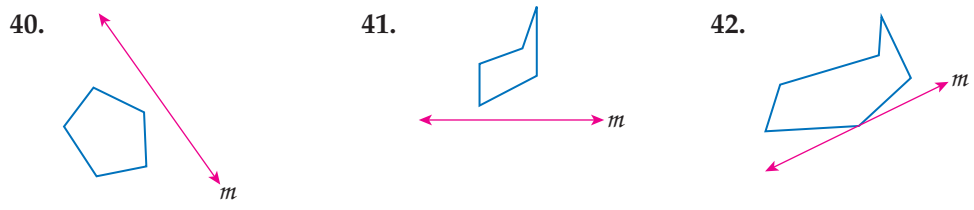
A **glide reflection** is a composition of a translation and a reflection in a line parallel to the direction of the translation.

- Is a glide reflection an *isometry*? Explain.
- Triangle DEF has vertices $D(4, 3)$, $E(2, -2)$ and $F(0, 1)$. Sketch the image of a glide reflection composed of the translation $(x, y) \rightarrow (x, y - 2)$ and a reflection in the y -axis.
- Triangle ABC has vertices $A(-3, -2)$, $B(-1, -3)$ and $C(2, -1)$. Sketch the image of a glide reflection composed of the translation $(x, y) \rightarrow (x + 3, y)$ and a reflection in $y = 1$.

38. Triangle XYZ with vertices X(5, 4), Y(3, -1), and Z(0, 2) is translated so that X' is at (3, 1). State the coordinates of Y' and Z'.
- (A) Y'(5, 2) and Z'(2, 5) (B) Y'(0, -3) and Z'(-3, 0)
 (C) Y'(1, -4) and Z'(-2, -1) (D) Y'(11, 4) and Z'(8, 6)
39. **ALGEBRA** Find the slope of a line through P(-2, 5) and T(2, -1).
- (A) $-\frac{3}{2}$ (B) -1 (C) 0 (D) $\frac{2}{3}$

Maintain Your Skills

Mixed Review Copy each figure. Draw the reflected image of each figure in line *m*. (Lesson 9-1)

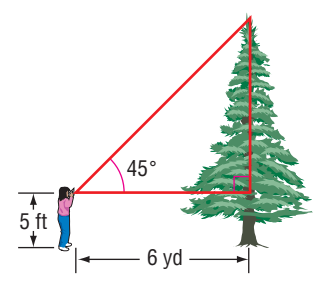


Name the missing coordinates for each quadrilateral. (Lesson 8-7)

43. QRST is an isosceles trapezoid. 44. ABCD is a parallelogram.



45. **LANDSCAPING** Juanna is a landscaper. She wishes to determine the height of a tree. Holding a drafter's 45° triangle so that one leg is horizontal, she sights the top of the tree along the hypotenuse as shown at the right. If she is 6 yards from the tree and her eyes are 5 feet from the ground, find the height of the tree. (Lesson 7-3)



State the assumption you would make to start an indirect proof of each statement. (Lesson 5-3)

46. Every shopper that walks through the door is greeted by a salesperson.
 47. If you get a job, you have filled out an application.
 48. If $4y + 17 \leq 41$, $y \leq 6$.
 49. If two lines are cut by a transversal and a pair of alternate interior angles are congruent, then the two lines are parallel.

Find the distance between each pair of parallel lines. (Lesson 3-6)

50. $x = -2$ 51. $y = -6$ 52. $y = 2x + 3$ 53. $y = x + 2$
 $x = 5$ $y = -1$ $y = 2x - 7$ $y = x - 4$

**Getting Ready for
the Next Lesson**

PREREQUISITE SKILL Use a protractor and draw an angle for each of the following degree measures. (To review drawing angles, see Lesson 1-4.)

54. 30 55. 45 56. 52 57. 60 58. 105 59. 150

9-3 Rotations

What You'll Learn

- Draw rotated images using the angle of rotation.
- Identify figures with rotational symmetry.

Vocabulary

- rotation
- center of rotation
- angle of rotation
- rotational symmetry
- direct isometry
- indirect isometry

How do some amusement rides illustrate rotations?

In 1926, Herbert Sellner invented the Tilt-A-Whirl. Today, no carnival is considered complete without these cars that send riders tipping and spinning as they make their way around a circular track. Manufactured in Faribault, Minnesota, the Tilt-A-Whirl provides an example of rotation.



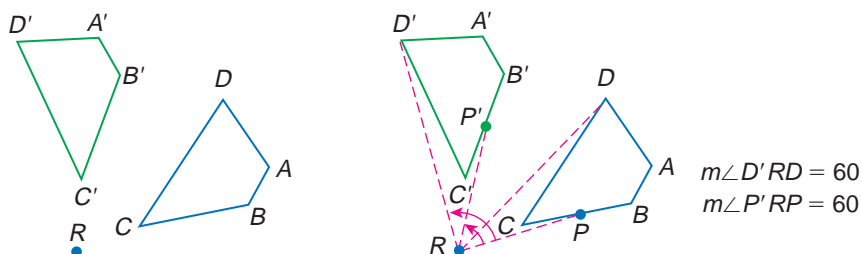
Study Tip

Turns

A rotation, sometimes called a turn, is generally measured as a counterclockwise turn. A half-turn is 180° and a full turn is 360° .

DRAW ROTATIONS A **rotation** is a transformation that turns every point of a preimage through a specified angle and direction about a fixed point. The fixed point is called the **center of rotation**.

In the figure, R is the center of rotation for the preimage $ABCD$. The measures of angles ARA' , BRB' , CRC' , and DRD' are equal. Any point P on the preimage of $ABCD$ has an image P' on $A'B'C'D'$ such that the measure of $\angle PRP'$ is a constant measure. This is called the **angle of rotation**.

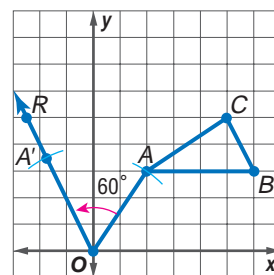


A rotation exhibits all of the properties of isometries, including preservation of distance and angle measure. Therefore, it is an isometry.

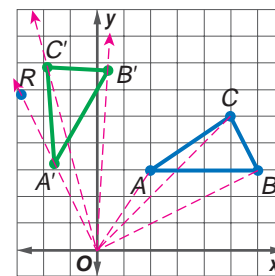
Example 1 Draw a Rotation

Triangle ABC has vertices $A(2, 3)$, $B(6, 3)$, and $C(5, 5)$. Draw the image of $\triangle ABC$ under a rotation of 60° counterclockwise about the origin.

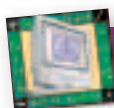
- First graph $\triangle ABC$.
- Draw a segment from the origin O , to point A .
- Use a protractor to measure a 60° angle counterclockwise with \overline{OA} as one side.
- Draw \overline{OR} .
- Use a compass to copy \overline{OA} onto \overline{OR} . Name the segment $\overline{OA'}$.



- Repeat with points B and C .
 $\triangle A'B'C'$ is the image of $\triangle ABC$ under a 60° counterclockwise rotation about the origin.



Another way to perform rotations is by reflecting a figure successively in two intersecting lines. Reflecting a figure once and then reflecting the image in a second line is another example of a composition of reflections.



Geometry Software Investigation

Reflections in Intersecting Lines

Construct a Figure

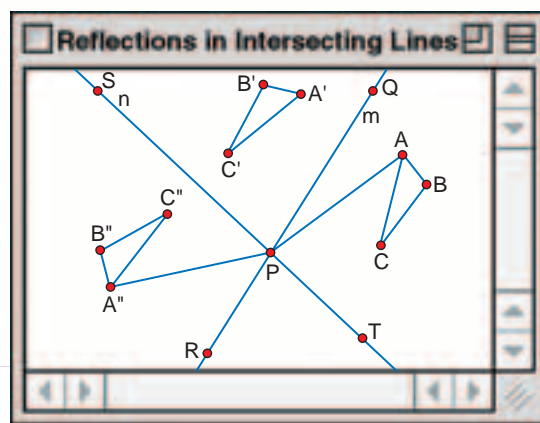
- Use The Geometer's Sketchpad to construct scalene triangle ABC .
- Construct lines m and n so that they intersect outside $\triangle ABC$.
- Label the point of intersection P .

Analyze

1. Reflect $\triangle ABC$ in line m . Then, reflect $\triangle A'B'C'$ in line n .
2. Describe the transformation of $\triangle ABC$ to $\triangle A''B''C''$.
3. Measure the acute angle formed by m and n .
4. Construct a segment from A to P and from A'' to P . Find the measure of the angle of rotation, $\angle APA''$.
5. Find $m\angle BPB''$ and $m\angle CPC''$.

Make a Conjecture

6. What is the relationship between the measures of the angles of rotation and the measure of the acute angle formed by m and n ?



When rotating a figure by reflecting it in two intersecting lines, there is a relationship between the angle of rotation and the angle formed by the intersecting lines.

Key Concept

Postulate 9.1 In a given rotation, if A is the preimage, A'' is the image, and P is the center of rotation, then the measure of the angle of rotation $\angle APA''$ is twice the measure of the acute or right angle formed by the intersecting lines of reflection.

Corollary 9.1 Reflecting an image successively in two perpendicular lines results in a 180° rotation.



Study Tip

Common Misconception

The order in which you reflect a figure in two nonperpendicular intersecting lines produces rotations of the same degree measure, but one is clockwise and the other is counterclockwise.

Example 2 Reflections in Intersecting Lines

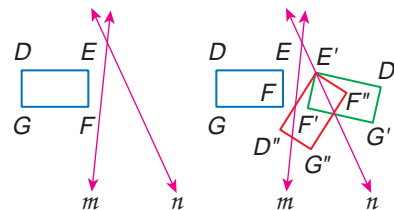
Find the image of rectangle $DEFG$ under reflections in line m and then line n .

First reflect rectangle $DEFG$ in line m .

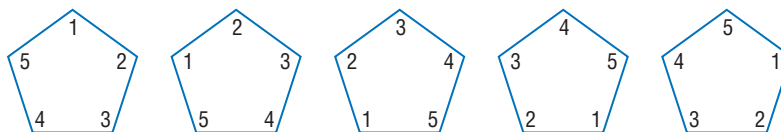
Then label the image $D'E'F'G'$.

Next, reflect the image in line n .

Rectangle $D''E''F''G''$ is the image of rectangle $DEFG$ under reflections in lines m and n . How can you transform $DEFG$ directly to $D''E''F''G''$ by using a rotation?



ROTATIONAL SYMMETRY Some objects have rotational symmetry. If a figure can be rotated less than 360 degrees about a point so that the image and the preimage are indistinguishable, then the figure has **rotational symmetry**.



In the figure, the pentagon has rotational symmetry of order 5 because there are 5 rotations of less than 360° (including 0 degrees) that produce an image indistinguishable from the original. The rotational symmetry has a *magnitude* of 72° because 360 degrees divided by the order, in this case 5, produces the magnitude of the symmetry.

Example 3 Identifying Rotational Symmetry

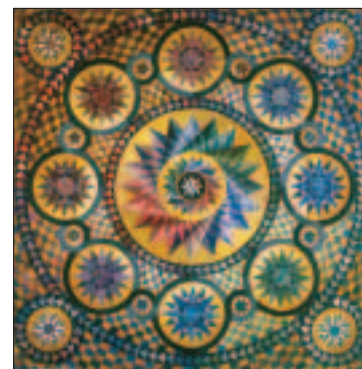
QUILTS One example of rotational symmetry artwork is quilt patterns. A quilt made by Judy Mathieson of Sebastopol, California, won the Masters Award for Contemporary Craftsmanship at the International Quilt Festival in 1999. Identify the order and magnitude of the symmetry in each part of the quilt.

a. large star in center of quilt

The large star in the center of the quilt has rotational symmetry of order 20 and magnitude of 18° .

b. entire quilt

The entire quilt has rotational symmetry of order 4 and magnitude of 90° .



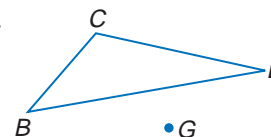
Check for Understanding

Concept Check

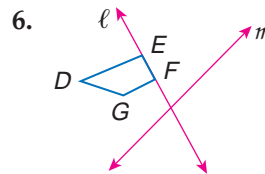
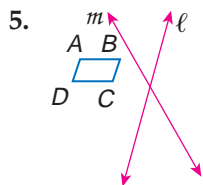
- OPEN ENDED** Draw a figure on the coordinate plane in Quadrant I. Rotate the figure clockwise 90 degrees about the origin. Then rotate the figure 90 degrees counterclockwise. Describe the results using the coordinates.
- Explain** two techniques that can be used to rotate a figure.
- Compare and contrast** translations and rotations.

Guided Practice

- Copy $\triangle BCD$ and rotate the triangle 60° counterclockwise about point G .



Copy each figure. Use a composition of reflections to find the rotation image with respect to lines ℓ and m .



- \overline{XY} has endpoints $X(-5, 8)$ and $Y(0, 3)$. Draw the image of \overline{XY} under a rotation of 45° clockwise about the origin.
- $\triangle PQR$ has vertices $P(-1, 8)$, $Q(4, -2)$, and $R(-7, -4)$. Draw the image of $\triangle PQR$ under a rotation of 90° counterclockwise about the origin.
- Identify the order and magnitude of the rotational symmetry in a regular hexagon.
- Identify the order and magnitude of the rotational symmetry in a regular octagon.

Application

- FANS** The blades of a fan exhibit rotational symmetry. Identify the order and magnitude of the symmetry of the blades of each fan in the pictures.



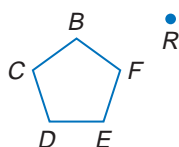
Practice and Apply

Homework Help

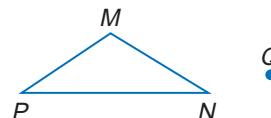
For Exercises	See Examples
12–15, 27	1
19–26	2
16–28, 28	3

Extra Practice
See page 772.

- Copy pentagon $BCDEF$. Then rotate the pentagon 110° counterclockwise about point R .



- Copy $\triangle MNP$. Then rotate the triangle 180° counterclockwise around point Q .



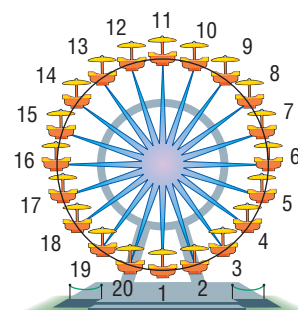
COORDINATE GEOMETRY Draw the rotation image of each figure 90° in the given direction about the center point and label the coordinates.

- $\triangle XYZ$ with vertices $X(0, -1)$, $Y(3,1)$, and $Z(1, 5)$ counterclockwise about the point $P(-1, 1)$
- $\triangle RST$ with vertices $R(0, 1)$, $S(5,1)$, and $T(2, 5)$ clockwise about the point $P(-2, 5)$

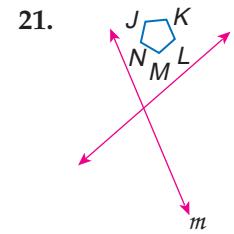
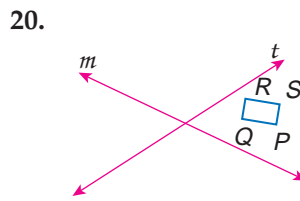
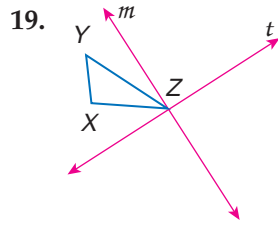
RECREATION For Exercises 16–18, use the following information.

A Ferris wheel's motion is an example of a rotation. The Ferris wheel shown has 20 cars.

- Identify the order and magnitude of the symmetry of a 20-seat Ferris wheel.
- What is the measure of the angle of rotation if seat 1 of a 20-seat Ferris wheel is moved to the seat 5 position?
- If seat 1 of a 20-seat Ferris wheel is rotated 144° , find the original seat number of the position it now occupies.



Copy each figure. Use a composition of reflections to find the rotation image with respect to lines m and t .



COORDINATE GEOMETRY Draw the rotation image of each triangle by reflecting the triangles in the given lines. State the coordinates of the rotation image and the angle of rotation.

22. $\triangle TUV$ with vertices $T(4, 0)$, $U(2, 3)$, and $V(1, 2)$, reflected in the y -axis and then the x -axis

23. $\triangle KLM$ with vertices $K(5, 0)$, $L(2, 4)$, and $M(-2, 4)$, reflected in the line $y = x$ and then the x -axis

24. $\triangle XYZ$ with vertices $X(5, 0)$, $Y(3, 4)$, and $Z(-3, 4)$, reflected in the line $y = -x$ and then the line $y = x$

25. **COORDINATE GEOMETRY** The point at $(2, 0)$ is rotated 30° counterclockwise about the origin. Find the exact coordinates of its image.

26. **MUSIC** A five-disc CD changer rotates as each CD is played. Identify the magnitude of the rotational symmetry as the changer moves from one CD to another.

More About . . .

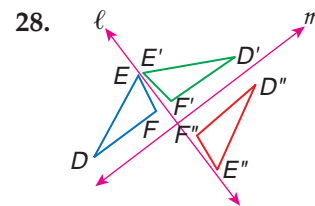
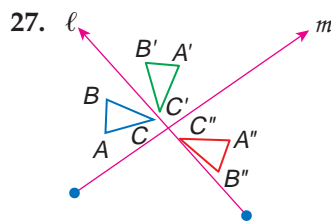


Music

Some manufacturers now make CD changers that are referred to as CD jukeboxes. These can hold hundreds of CDs, which may be more than a person's entire CD collection.

Source: www.usatoday.com

Determine whether the indicated composition of reflections is a rotation. Explain.

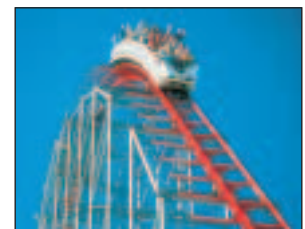


AMUSEMENT RIDES For each ride, determine whether the rider undergoes a rotation. Write *yes* or *no*.

29. spinning teacups

30. scrambler

31. roller coaster loop

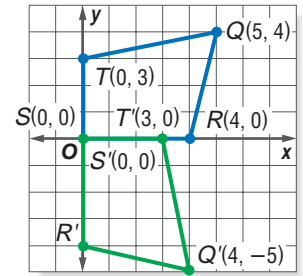


32. **ALPHABET** Which capital letters of the alphabet produce the same letter after being rotated 180° ?

33. **CRITICAL THINKING** In $\triangle ABC$, $m\angle BAC = 40$. Triangle $AB'C$ is the image of $\triangle ABC$ under reflection and $\triangle AB'C'$ is the image of $\triangle AB'C$ under reflection. How many such reflections would be necessary to map $\triangle ABC$ onto itself?

34. **CRITICAL THINKING** If a rotation is performed on a coordinate plane, what angles of rotation would make the rotations easier? Explain.

35. **COORDINATE GEOMETRY** Quadrilateral $QRST$ is rotated 90° clockwise about the origin. Describe the transformation using coordinate notation.
36. Triangle FGH is rotated 80° clockwise and then rotated 150° counterclockwise about the origin. To what counterclockwise rotation about the origin is this equivalent?



- CRITICAL THINKING** For Exercises 37–39, use the following information. Points that do not change position under a transformation are called *invariant points*. For each of the following transformations, identify any invariant points.
37. reflection in a line
38. a rotation of x° ($0 < x < 360$) about point P
39. $(x, y) \rightarrow (x + a, y + b)$, where a and b are not 0

40. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

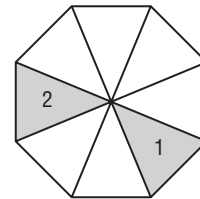
How do amusement rides exemplify rotations?

Include the following in your answer:

- a description of how the Tilt-A-Whirl actually rotates two ways, and
- other amusement rides that use rotation.



41. In the figure, describe the rotation that moves triangle 1 to triangle 2.
- (A) 180° clockwise
- (B) 135° clockwise
- (C) 135° counterclockwise
- (D) 90° counterclockwise



42. **ALGEBRA** Suppose x is $\frac{2}{5}$ of y and y is $\frac{1}{3}$ of z . If $x = 6$, then $z = ?$
- (A) $\frac{4}{5}$ (B) $\frac{18}{5}$ (C) 5 (D) 45

Extending the Lesson

A **direct isometry** is one in which the image of a figure is found by moving the figure intact within the plane. An **indirect isometry** cannot be performed by maintaining the orientation of the points as in a direct isometry.

43. Copy and complete the table below. Determine whether each transformation preserves the given property. Write *yes* or *no*.

Transformation	angle measure	betweenness of points	orientation	collinearity	distance measure
reflection					
translation					
rotation					

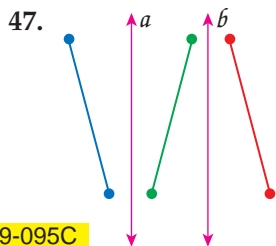
Identify each type of transformation as a direct isometry or an indirect isometry.

44. reflection 45. translation 46. rotation

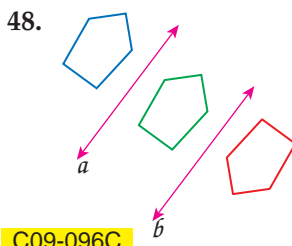


Maintain Your Skills

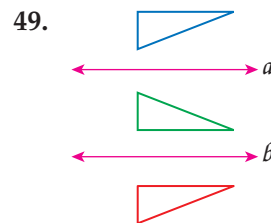
Mixed Review In each figure, $a \parallel b$. Determine whether the blue figure is a translation image of the red figure. Write *yes* or *no*. Explain your answer. (Lesson 9-2)



C09-095C



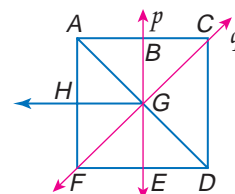
C09-096C



C09-097C

For the figure to the right, name the reflected image of each image. (Lesson 9-1)

50. \overline{AG} across line p
51. F across point G
52. \overline{GE} across line q
53. $\angle CGD$ across line p

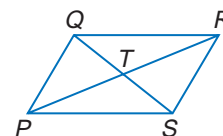


C09-098C

Complete each statement about $\square PQRS$.

Justify your answer. (Lesson 8-2)

54. $\overline{QR} \parallel$?
55. $\overline{PT} \cong$?
56. $\angle SQR \cong$?
57. $\angle QPS \cong$?



C09-099C

58. **SURVEYING** A surveyor is 100 meters from a building and finds that the angle of elevation to the top of the building is 23° . If the surveyor's eye level is 1.55 meters above the ground, find the height of the building. (Lesson 7-5)

Determine whether the given measures can be the lengths of the sides of a triangle. Write *yes* or *no*. (Lesson 5-4)

59. 6, 8, 16 60. 12, 17, 20 61. 22, 23, 37

Getting Ready for the Next Lesson

PREREQUISITE SKILL Find whole number values for the variable so each equation is true. (To review solving problems by making a table, see pages 737-738.)

62. $180a = 360$
63. $180a + 90b = 360$
64. $135a + 45b = 360$
65. $120a + 30b = 360$
66. $180a + 60b = 360$
67. $180a + 30b = 360$

Practice Quiz 1

Lesson 9-1 through 9-3

Graph each figure and the image in the given reflection. (Lesson 9-1)

1. $\triangle DEF$ with vertices $D(-1, 1)$, $E(1, 4)$ and $F(3, 2)$ in the origin
2. quadrilateral $ABCD$ with vertices $A(0, 2)$, $B(2, 2)$, $C(3, 0)$, and $D(-1, 1)$ in the line $y = x$

Graph each figure and the image under the given translation. (Lesson 9-2)

3. \overline{PQ} with endpoints $P(1, -4)$ and $Q(4, -1)$ under the translation left 3 units and up 4 units
4. $\triangle KLM$ with vertices $K(-2, 0)$, $L(-4, 2)$, and $M(0, 4)$ under the translation $(x, y) \rightarrow (x + 1, y - 4)$

5. Identify the order and magnitude of the symmetry of a 36-horse carousel. (Lesson 9-3)

9-4 Tessellations

What You'll Learn

- Identify regular tessellations.
- Create tessellations with specific attributes.

Vocabulary

- tessellation
- regular tessellation
- uniform
- semi-regular tessellation

How are tessellations used in art?

M.C. Escher (1898–1972) was a Dutch graphic artist famous for repeating geometric patterns. He was also well known for his spatial illusions, impossible buildings, and techniques in woodcutting and lithography. In the picture, figures can be reduced to basic regular polygons. Equilateral triangles and regular hexagons are prominent in the repeating patterns.



Symmetry Drawing E103,
M.C. Escher

©2002 Cordon Art B.V., Baarn, Holland. All rights reserved.

Study Tip

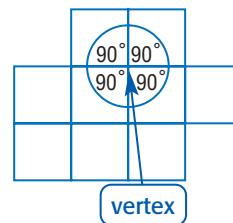
Reading Math

The word *tessellation* comes from the Latin word *tessera* which means “a square tablet.” These small square pieces of stone or tile were used in mosaics.

REGULAR TESSELLATIONS Reflections, translations, and rotations can be used to create patterns using polygons. A pattern that covers a plane by transforming the same figure or set of figures so that there are no overlapping or empty spaces is called a **tessellation**.

In a tessellation, the sum of the measures of the angles of the polygons surrounding a point (at a vertex) is 360.

You can use what you know about angle measures in regular polygons to help determine which polygons tessellate.



Geometry Activity

Tessellations of Regular Polygons

Model and Analyze

- Study a set of pattern blocks to determine which shapes are regular.
 - Make a tessellation with each type of regular polygon.
1. Which shapes in the set are regular?
 2. Write an expression showing the sum of the angles at each vertex of the tessellation.
 3. Copy and complete the table below.

Regular Polygon	triangle	square	pentagon	hexagon	heptagon	octagon
Measure of One Interior Angle						
Does it tessellate?						

Make a Conjecture

4. What must be true of the angle measure of a regular polygon for a regular tessellation to occur?

The tessellations you formed in the Geometry Activity are regular tessellations. A **regular tessellation** is a tessellation formed by only one type of regular polygon. In the activity, you found that if a regular polygon has an interior angle with a measure that is a factor of 360, then the polygon will tessellate the plane.

Example 1 Regular Polygons

Determine whether a regular 24-gon tessellates the plane. Explain.

Let $\angle 1$ represent one interior angle of a regular 24-gon.

$$\begin{aligned} m\angle 1 &= \frac{180(n-2)}{n} && \text{Interior Angle Formula} \\ &= \frac{180(24-2)}{24} && \text{Substitution} \\ &= 165 && \text{Simplify.} \end{aligned}$$

Since 165 is not a factor of 360, a 24-gon will not tessellate the plane.

Study Tip

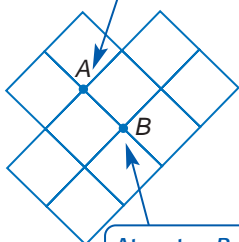
Look Back

To review finding the measure of an interior angle of a regular polygon, see Lesson 8-1.

TESSELLATIONS WITH SPECIFIC ATTRIBUTES A tessellation pattern can contain any type of polygon. Tessellations containing the same arrangement of shapes and angles at each vertex are called **uniform**.

uniform

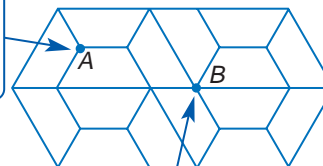
At vertex A , there are four congruent angles.



At vertex B , there are the same four congruent angles.

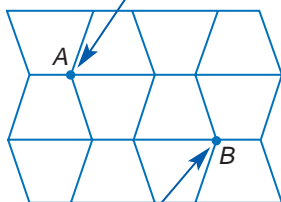
not uniform

At vertex A , there are three angles that are all congruent.



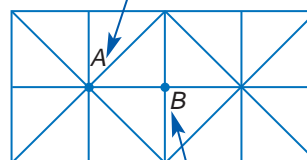
At vertex B , there are five angles; four are congruent and one is different.

At vertex A , there are four angles that consist of two congruent pairs.



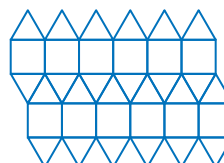
At vertex B , there are the same two congruent pairs.

At vertex A , there are eight congruent angles.



At vertex B , there are four congruent angles.

Tessellations can be formed using more than one type of polygon. A uniform tessellation formed using two or more regular polygons is called a **semi-regular tessellation**.



Study Tip

Drawing

When creating your own tessellation, it is sometimes helpful to complete one pattern piece, cut it out, and trace it for the other units.

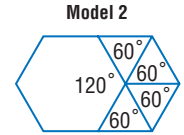
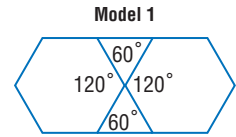
Example 2 Semi-Regular Tessellation

Determine whether a semi-regular tessellation can be created from regular hexagons and equilateral triangles, all having sides 1 unit long.

Method 1 Make a model.

Two semi-regular models are shown. You will notice that the spaces at each vertex can be filled in with equilateral triangles. Model 1 has two hexagons and two equilateral triangles arranged in an alternating pattern around each vertex.

Model 2 has one hexagon and four equilateral triangles at each vertex.



Method 2 Solve algebraically.

Each interior angle of a regular hexagon measures $\frac{180(6-2)}{6}$ or 120° .

Each angle of an equilateral triangle measures 60° . Find whole-number values for h and t so that $120h + 60t = 360$.

Let $h = 1$.

$$120(1) + 60t = 360$$

$$120 + 60t = 360$$

$$60t = 240$$

$$t = 4$$

Substitution

Simplify.

Subtract from each side.

Divide each side by 60.

Let $h = 2$.

$$120(2) + 60t = 360$$

$$240 + 60t = 360$$

$$60t = 120$$

$$t = 2$$

When $h = 1$ and $t = 4$, there is one hexagon with four equilateral triangles at each vertex. (Model 2)

When $h = 2$ and $t = 2$, there are two hexagons and two equilateral triangles. (Model 1)

Note if $h = 0$ and $t = 6$ or $h = 3$ and $t = 0$, then the tessellations are regular because there would be only one regular polygon.

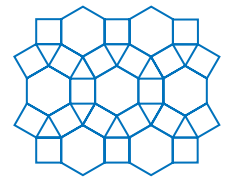
Example 3 Classify Tessellations

FLOORING Tile flooring comes in many shapes and patterns. Determine whether the pattern is a tessellation.

If so, describe it as *uniform*, *not uniform*, *regular*, or *semi-regular*.

The pattern is a tessellation because at the different vertices the sum of the angles is 360° .

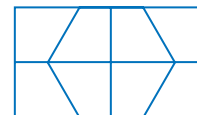
The tessellation is uniform because at each vertex there are two squares, a triangle, and a hexagon arranged in the same order. The tessellation is also semi-regular since more than one regular polygon is used.



Check for Understanding

Concept Check

- Compare and contrast a semi-regular tessellation and a uniform tessellation.
- OPEN ENDED** Use these pattern blocks that are 1 unit long on each side to create a tessellation.
- Explain why the tessellation is *not* a regular tessellation.



Guided Practice Determine whether each regular polygon tessellates the plane. Explain.

4. decagon

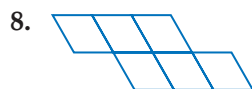
5. 30-gon

Determine whether a semi-regular tessellation can be created from each set of figures. Assume that each figure has a side length of 1 unit.

6. a square and a triangle

7. an octagon and a square

Determine whether each pattern is a tessellation. If so, describe it as *uniform*, *not uniform*, *regular*, or *semi-regular*.



Application 10. **QUILTING** The “Postage Stamp” pattern can be used in quilting. Explain why this is a tessellation and what kind it is.



Practice and Apply

Homework Help

For Exercises	See Examples
11–16	1
17–20	2
21–28	3

Extra Practice
See page 772.

Determine whether each regular polygon tessellates the plane. Explain.

11. nonagon

12. hexagon

13. equilateral triangle

14. dodecagon

15. 23-gon

16. 36-gon

Determine whether a semi-regular tessellation can be created from each set of figures. Assume that each figure has a side length of 1 unit.

17. regular octagons and rhombi

18. regular dodecagons and equilateral triangles

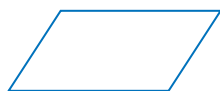
19. regular dodecagons, squares, and equilateral triangles

20. regular heptagons, squares, and equilateral triangles

Determine whether each polygon tessellates the plane. If so, describe the tessellation as *uniform*, *not uniform*, *regular*, or *semi-regular*.

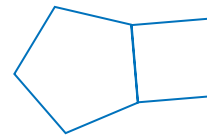
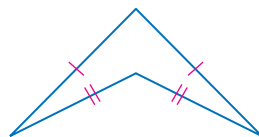
21. parallelogram

22. kite



23. quadrilateral

24. pentagon and square



Determine whether each pattern is a tessellation. If so, describe it as *uniform*, *not uniform*, *regular*, or *semi-regular*.



Career Choices



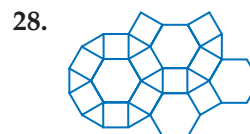
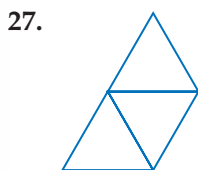
Bricklayer

Bricklayers arrange and secure bricks, concrete blocks, tiles, and other building materials to construct or repair walls, partitions, arches, fireplaces, and other structures. They must understand how the pieces tessellate to complete the structure specifications.

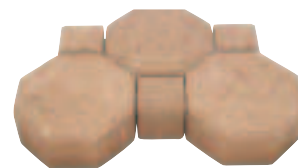
Online Research

For information about a career as a bricklayer, visit:

www.geometryonline.com/careers



29. **BRICKWORK** In the picture, suppose the side of the octagon is the same length as the side of the square. What kind of tessellation is this?

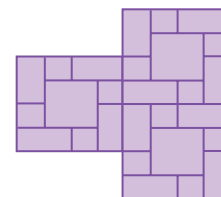


Determine whether each statement is *always*, *sometimes*, or *never* true. Justify your answers.

30. Any triangle will tessellate the plane.
31. Semi-regular tessellations are not uniform.
32. Uniform tessellations are semi-regular.
33. Every quadrilateral will tessellate the plane.
34. Regular 16-gons will tessellate the plane.

INTERIOR DESIGN For Exercises 35 and 36, use the following information.

Kele's family is tiling the entry floor with the tiles in the pattern shown.



35. Determine whether the pattern is a tessellation.
36. Is the tessellation *uniform*, *regular*, or *semi-regular*?

37. **BEES** A honeycomb is composed of hexagonal cells made of wax in which bees store honey. Determine whether this pattern is a tessellation. If so, describe it as *uniform*, *not uniform*, *regular*, or *semi-regular*.



38. **CRITICAL THINKING** What could be the measures of the interior angles in a pentagon that tessellate a plane? Is this tessellation regular? Is it uniform?

39. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are tessellations used in art?

Include the following in your answer:

- how equilateral triangles and regular hexagons form a tessellation, and
- other geometric figures that can be found in the picture.

40. Find the measure of an interior angle of a regular nonagon.

(A) 150 (B) 147 (C) 140 (D) 115

41. **ALGEBRA** Evaluate $\frac{360(x-2)}{2x} - \frac{180}{x}$ if $x = 12$.

(A) 135 (B) 150 (C) 160 (D) 225

Standardized Test Practice

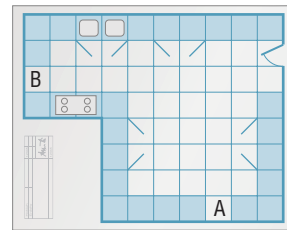
(A) (B) (C) (D)

Maintain Your Skills

Mixed Review **COORDINATE GEOMETRY** Draw the rotation image of each figure 90° in the given direction about the center point and label the coordinates. (Lesson 9-3)

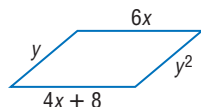
42. $\triangle ABC$ with $A(8, 1)$, $B(2, -6)$, and $C(-4, -2)$ counterclockwise about $P(-2, 2)$
 43. $\triangle DEF$ with $D(6, 2)$, $E(6, -3)$, and $F(2, 3)$ clockwise about $P(3, -2)$
 44. $\square GHIJ$ with $G(-1, 2)$, $H(-3, -3)$, $I(-5, -6)$, and $J(-3, -1)$ counterclockwise about $P(-2, -3)$
 45. rectangle $KLMN$ with $K(-3, -5)$, $L(3, 3)$, $M(7, 0)$, and $N(1, -8)$ counterclockwise about $P(-2, 0)$

46. **REMODELING** The diagram at the right shows the floor plan of Justin's kitchen. Each square on the diagram represents a 3 foot by 3 foot area. While remodeling his kitchen, Justin moved his refrigerator from square A to square B. Describe the move. (Lesson 9-2)

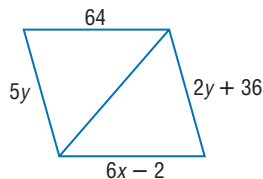


ALGEBRA Find x and y so that each quadrilateral is a parallelogram. (Lesson 8-3)

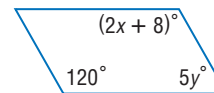
47.



48.



49.



Determine whether each set of measures are the sides of a right triangle. Then state whether they form a Pythagorean triple. (Lesson 7-2)

50. 12, 16, 20

51. 9, 10, 15

52. 2.5, 6, 6.5

53. $14, 14\sqrt{3}, 28$

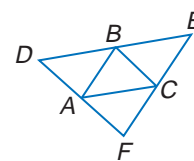
54. 14, 48, 50

55. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

Points A , B , and C are the midpoints of \overline{DF} , \overline{DE} , and \overline{EF} , respectively. (Lesson 6-4)

56. If $BC = 11$, $AC = 13$, and $AB = 15$, find the perimeter of $\triangle DEF$.

57. If $DE = 18$, $DA = 10$, and $FC = 7$, find AB , BC , and AC .



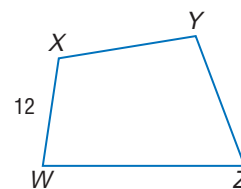
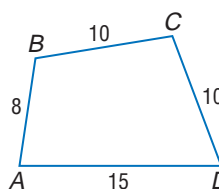
COORDINATE GEOMETRY For Exercises 58–61, use the following information. The vertices of quadrilateral $PQRS$ are $P(5, 2)$, $Q(1, 6)$, $R(-3, 2)$, and $S(1, -2)$. (Lesson 3-3)

58. Show that the opposite sides of quadrilateral $PQRS$ are parallel.
 59. Show that the adjacent sides of quadrilateral $PQRS$ are perpendicular
 60. Determine the length of each side of quadrilateral $PQRS$.
 61. What type of figure is quadrilateral $PQRS$?

Getting Ready for the Next Lesson

PREREQUISITE SKILL If quadrilateral $ABCD \sim$ quadrilateral $WXYZ$, find each of the following. (To review similar polygons, see Lesson 6-2.)

62. scale factor of $ABCD$ to $WXYZ$
 63. XY
 64. YZ
 65. WZ





Geometry Activity

A Follow-up of Lesson 9-4

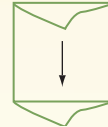
Tessellations and Transformations

Activity 1 Make a tessellation using a translation.

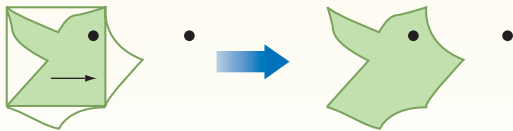
Step 1 Start by drawing a square. Then copy the figure shown below.



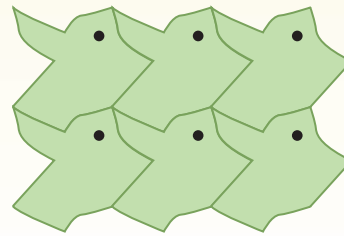
Step 2 Translate the shape on the top side to the bottom side.



Step 3 Translate the figure on the left side and the dot to the right side to complete the pattern.



Step 4 Repeat this pattern on a tessellation of squares.



Activity 2 Make a tessellation using a rotation.

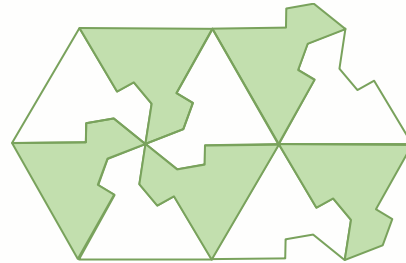
Step 1 Start by drawing an equilateral triangle. Then draw a trapezoid inside the right side of the triangle.



Step 2 Rotate the trapezoid so you can copy the change on the side indicated.

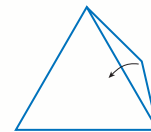


Step 3 Repeat this pattern on a tessellation of equilateral triangles. Alternating colors can be used to best show the tessellation.

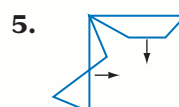
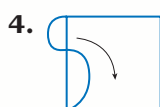
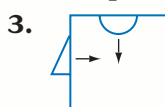


Model and Analyze

1. Is the area of the square in Step 1 of Activity 1 the same as the area of the new shape in Step 2? Explain.
2. Describe how you would create the unit for the pattern shown at the right.



Make a tessellation for each pattern described. Use a tessellation of two rows of three squares as your base.



9-5 Dilations

What You'll Learn

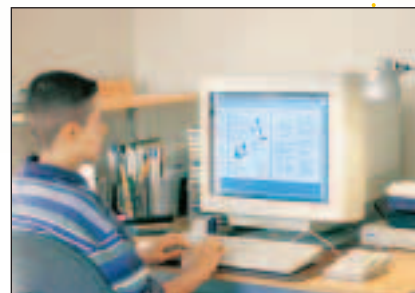
- Determine whether a dilation is an enlargement, a reduction, or a congruence transformation.
- Determine the scale factor for a given dilation.

Vocabulary

- dilation
- similarity transformation

How do you use dilations when you use a computer?

Have you ever tried to paste an image into a word processing document and the picture was too large? Many word processing programs allow you to scale the size of the picture so that you can fit it in your document. Scaling a picture is an example of a dilation.

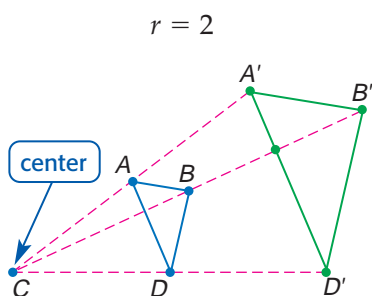


Study Tip

Scale Factor

When discussing dilations, scale factor has the same meaning as with proportions. The letter r usually represents the scale factor.

CLASSIFY DILATIONS All of the transformations you have studied so far in this chapter produce images that are congruent to the original figure. A dilation is another type of transformation. However, the image of a dilation may be a different size than the original figure. A **dilation** is a transformation that changes the size of a figure. A dilation requires a center point and a scale factor. The figures below show how dilations can result in a larger figure and a smaller figure than the original.



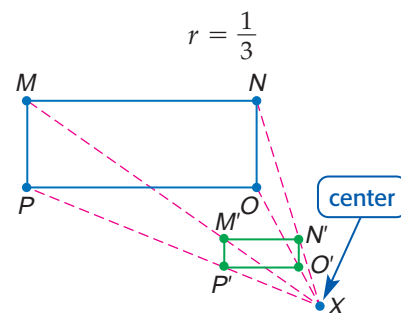
Triangle $A'B'D'$ is a dilation of $\triangle ABD$.

$$CA' = 2(CA)$$

$$CB' = 2(CB)$$

$$CD' = 2(CD)$$

$\triangle A'B'D'$ is larger than $\triangle ABD$.



Rectangle $M'N'O'P'$ is a dilation of rectangle $MNOP$.

$$XM' = \frac{1}{3}(XM) \quad XN' = \frac{1}{3}(XN)$$

$$XO' = \frac{1}{3}(XO) \quad XP' = \frac{1}{3}(XP)$$

Rectangle $M'N'O'P'$ is smaller than rectangle $MNOP$.

The value of r determines whether the dilation is an enlargement or a reduction.

Key Concept

Dilation

If $|r| > 1$, the dilation is an enlargement.

If $0 < |r| < 1$, the dilation is a reduction.

If $|r| = 1$, the dilation is a congruence transformation.

Study Tip

Isometry Dilation

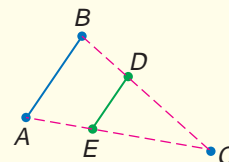
A dilation with a scale factor of 1 produces an image that coincides with the preimage. The two are congruent.

As you can see in the figures on the previous page, dilation preserves angle measure, betweenness of points, and collinearity, but does not preserve distance. That is, dilations produce similar figures. Therefore, a dilation is a **similarity transformation**.

This means that $\triangle ABD \sim \triangle A'B'D'$ and $\square MNOP \sim \square M'N'O'P'$. This implies that $\frac{A'B'}{AB} = \frac{B'D'}{BD} = \frac{A'D'}{AD}$ and $\frac{M'N'}{MN} = \frac{N'O'}{NO} = \frac{O'P'}{OP} = \frac{M'P'}{MP}$. The ratios of measures of the corresponding parts is equal to the absolute value scale factor of the dilation, $|r|$. So, $|r|$ determines the size of the image as compared to the size of the preimage.

Theorem 9.1

If a dilation with center C and a scale factor of r transforms A to E and B to D , then $ED = |r|(AB)$.



You will prove Theorem 9.1 in Exercise 41.

Example 1 Determine Measures Under Dilations

Find the measure of the dilation image $\overline{A'B'}$ or the preimage \overline{AB} using the given scale factor.

a. $AB = 12, r = -2$

$$A'B' = |r|(AB) \quad \text{Dilation Theorem}$$

$$A'B' = 2(12) \quad |r| = 2, AB = 12$$

$$A'B' = 24 \quad \text{Multiply.}$$

b. $A'B' = 36, r = \frac{1}{4}$

$$A'B' = |r|(AB) \quad \text{Dilation Theorem}$$

$$36 = \frac{1}{4}(AB) \quad A'B' = 36, |r| = \frac{1}{4}$$

$$144 = AB \quad \text{Multiply each side by 4.}$$

When the scale factor is negative, the image falls on the opposite side of the center than the preimage.

Key Concept

Dilations

If $r > 0$, P' lies on \overline{CP} , and $CP' = r \cdot CP$.

If $r < 0$, P' lies on $\overline{CP'}$ the ray opposite \overline{CP} , and $CP' = |r| \cdot CP$.

The center of a dilation is always its own image.

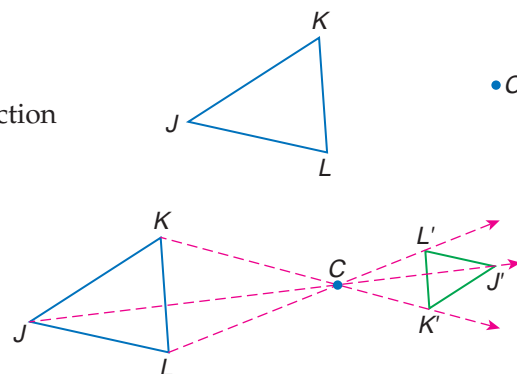
Example 2 Draw a Dilation

Draw the dilation image of $\triangle JKL$ with center C and $r = -\frac{1}{2}$.

Since $0 < |r| < 1$, the dilation is a reduction of $\triangle JKL$.

Draw \overline{CJ} , \overline{CK} , and \overline{CL} . Since r is negative, J' , K' , and L' will lie on $\overline{CJ'}$, $\overline{CK'}$, and $\overline{CL'}$, respectively. Locate J' , K' , and L' so that $CJ' = \frac{1}{2}(CJ)$, $CK' = \frac{1}{2}(CK)$, and $CL' = \frac{1}{2}(CL)$.

Draw $\triangle J'K'L'$.



In the coordinate plane, you can use the scale factor to determine the coordinates of the image of dilations centered at the origin.

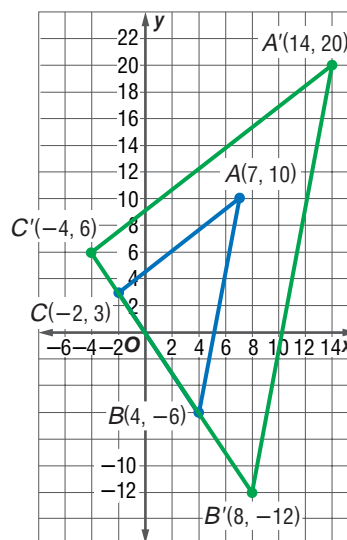
Theorem 9.2

If $P(x, y)$ is the preimage of a dilation centered at the origin with a scale factor r , then the image is $P'(rx, ry)$.

Example 3 Dilations in the Coordinate Plane

COORDINATE GEOMETRY Triangle ABC has vertices $A(7, 10)$, $B(4, -6)$, and $C(-2, 3)$. Find the image of $\triangle ABC$ after a dilation centered at the origin with a scale factor of 2. Sketch the preimage and the image.

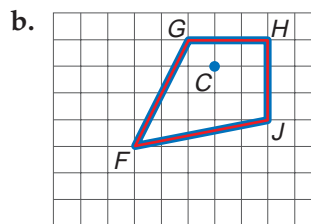
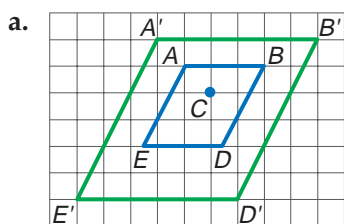
Preimage (x, y)	Image ($2x, 2y$)
$A(7, 10)$	$A'(14, 20)$
$B(4, -6)$	$B'(8, -12)$
$C(-2, 3)$	$C'(-4, 6)$



IDENTIFY THE SCALE FACTOR In Chapter 6, you found scale factors of similar figures. If you know the measurement of a figure and its dilated image, you can determine the scale factor.

Example 4 Identify Scale Factor

Determine the scale factor for each dilation with center C . Then determine whether the dilation is an *enlargement*, *reduction*, or *congruence transformation*.



$$\begin{aligned} \text{scale factor} &= \frac{\text{image length}}{\text{preimage length}} \\ &= \frac{6 \text{ units}}{3 \text{ units}} \quad \leftarrow \begin{array}{l} \text{image length} \\ \text{preimage length} \end{array} \\ &= 2 \quad \text{Simplify.} \end{aligned}$$

Since the scale factor is greater than 1, the dilation is an enlargement.

$$\begin{aligned} \text{scale factor} &= \frac{\text{image length}}{\text{preimage length}} \\ &= \frac{4 \text{ units}}{4 \text{ units}} \quad \leftarrow \begin{array}{l} \text{image length} \\ \text{preimage length} \end{array} \\ &= 1 \quad \text{Simplify.} \end{aligned}$$

Since the scale factor is 1, the dilation is a congruence transformation.

Study Tip

Look Back
To review **scale factor**, see Lesson 6-2.

Example 5 Scale Drawing

Multiple-Choice Test Item

Jacob wants to make a scale drawing of a painting in an art museum. The painting is 4 feet wide and 8 feet long. Jacob decides on a dilation reduction factor of $\frac{1}{6}$. What size paper will he need to make a complete sketch?

- (A) $8\frac{1}{2}$ in. by 11 in. (B) 9 in. by 12 in. (C) 11 in. by 14 in. (D) 11 in. by 17 in.

The
Princeton
Review

Test-Taking Tip

Compare Measurements
Compare the measurements given in the problem to those in the answers. These answers are in inches, so convert feet to inches before using the scale factor. It may make calculations easier.

Read the Test Item

The painting's dimensions are given in feet, and the paper choices are in inches. You need to convert from feet to inches in the problem.

Solve the Test Item

Step 1 Convert feet to inches.

$$4 \text{ feet} = 4(12) \text{ or } 48 \text{ inches}$$

$$8 \text{ feet} = 8(12) \text{ or } 96 \text{ inches}$$

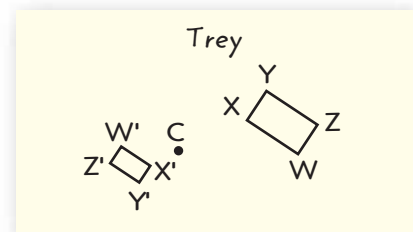
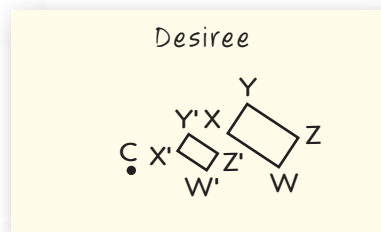
Step 2 Use the scale factor to find the image dimensions.

$$w = \frac{1}{6}(48) \text{ or } 8 \qquad \ell = \frac{1}{6}(96) \text{ or } 16$$

Step 3 The dimensions of the image are 8 inches by 16 inches. Choice D is the only size paper on which the scale drawing will fit.

Check for Understanding

- Concept Check**
- Find a counterexample to disprove the statement *All dilations are isometries*.
 - OPEN ENDED** Draw a figure on the coordinate plane. Then show a dilation of the figure that is a reduction and a dilation of the figure that is an enlargement.
 - FIND THE ERROR** Desiree and Trey are trying to describe the effect of a negative r value for a dilation of quadrilateral $WXYZ$.



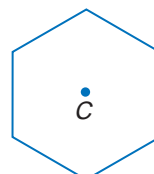
Who is correct? Explain your reasoning.

Guided Practice Draw the dilation image of each figure with center C and the given scale factor.

4. $r = 4$



5. $r = \frac{1}{5}$



6. $r = -2$



Find the measure of the dilation image $\overline{A'B'}$ or the preimage \overline{AB} using the given scale factor.

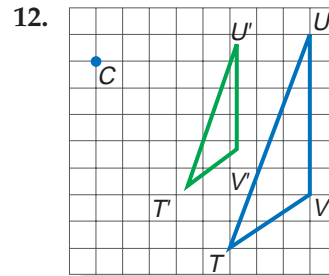
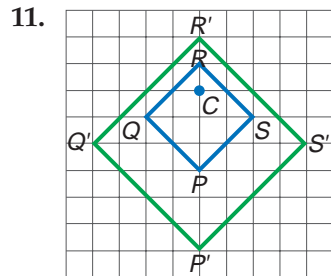
7. $AB = 3, r = 4$

8. $A'B' = 8, r = -\frac{2}{5}$

9. \overline{PQ} has endpoints $P(9, 0)$ and $Q(0, 6)$. Find the image of \overline{PQ} after a dilation centered at the origin with a scale factor $r = \frac{1}{3}$. Sketch the preimage and the image.

10. Triangle KLM has vertices $K(5, 8)$, $L(-3, 4)$, and $M(-1, -6)$. Find the image of $\triangle KLM$ after a dilation centered at the origin with scale factor of 3. Sketch the preimage and the image.

Determine the scale factor for each dilation with center C . Determine whether the dilation is an *enlargement*, *reduction*, or *congruence transformation*.



13. Alexis made a scale drawing of the plan for her spring garden. It will be a rectangle measuring 18 feet by 12 feet. On the scaled version, it measures 8 inches on the longer sides. What is the measure of each of the shorter sides?

(A) $6\frac{1}{2}$ in.

(B) $5\frac{1}{2}$ in.

(C) $5\frac{1}{3}$ in.

(D) $4\frac{3}{5}$ in.

Practice and Apply

Homework Help

For Exercises	See Examples
14–19	1
20–25	2
26–29	3
32–37	4
30–31, 38–40	5

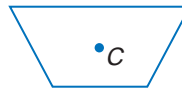
Extra Practice
See page 772.

Draw the dilation image of each figure with center C and the given scale factor.

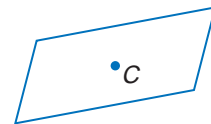
14. $r = 3$



15. $r = 2$



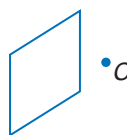
16. $r = \frac{1}{2}$



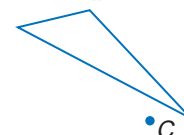
17. $r = \frac{2}{5}$



18. $r = -1$



19. $r = -\frac{1}{4}$



Find the measure of the dilation image $\overline{S'T'}$ or the preimage \overline{ST} using the given scale factor.

20. $ST = 6, r = -1$

21. $ST = \frac{4}{5}, r = \frac{3}{4}$

22. $S'T' = 12, r = \frac{2}{3}$

23. $S'T' = \frac{12}{5}, r = -\frac{3}{5}$

24. $ST = 32, r = -\frac{5}{4}$

25. $ST = 2.25, r = 0.4$

COORDINATE GEOMETRY Find the image of each polygon, given the vertices, after a dilation centered at the origin with a scale factor of 2. Then graph a dilation centered at the origin with a scale factor of $\frac{1}{2}$.

26. $F(3, 4), G(6, 10), H(-3, 5)$ 27. $X(1, -2), Y(4, -3), Z(6, -1)$
 28. $P(1, 2), Q(3, 3), R(3, 5), S(1, 4)$ 29. $K(4, 2), L(-4, 6), M(-6, -8), N(6, -10)$

Determine the scale factor for each dilation with center C . Determine whether the dilation is an *enlargement*, *reduction*, or *congruence transformation*.

- 30.
- 31.
- 32.
- 33.
- 34.
- 35.



More About . . .

Airplanes

The SR-71 Blackbird is 107 feet 5 inches long with a wingspan of 55 feet 7 inches and can fly at speeds over 2200 miles per hour. It can fly nonstop from Los Angeles to Washington, D.C., in just over an hour, while a standard commercial jet takes about five hours to complete the trip.

Source: NASA

36. **AIRPLANES** Etay is building a model of the SR-71 Blackbird. If the wingspan of his model is 14 inches, what is the approximate scale factor of the model?

PHOTOCOPY For Exercises 37 and 38, refer to the following information.

A 10-inch by 14-inch rectangular design is being reduced on a photocopier by a factor of 75%.

37. What are the new dimensions of the design?
 38. How has the area of the preimage changed?

For Exercises 39 and 40, use the following information.

A dilation on a rectangle has a scale factor of 4.

39. What is the effect of the dilation on the perimeter of the rectangle?
 40. What is the effect of the dilation on the area of the rectangle?

41. **PROOF** Write a paragraph proof of Theorem 9.1.

42. Triangle ABC has vertices $A(12, 4), B(4, 8),$ and $C(8, -8)$. After two successive dilations centered at the origin with the same scale factor, the final image has vertices $A''(3, 1), B''(1, 2),$ and $C''(2, -2)$. Determine the scale factor r of each dilation from $\triangle ABC$ to $\triangle A''B''C''$.

43. Segment XY has endpoints $X(4, 2)$ and $Y(0, 5)$. After a dilation, the image has endpoints of $X'(6, 3)$ and $Y'(12, 11)$. Find the absolute value of the scale factor.

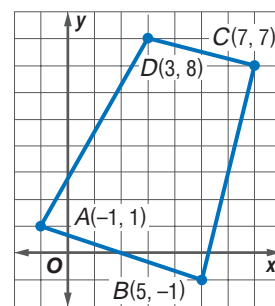
DIGITAL PHOTOGRAPHY For Exercises 44–46, use the following information.

Dinah is editing a digital photograph that is 640 pixels wide and 480 pixels high on her monitor.

44. If Dinah zooms the image on her monitor 150%, what are the dimensions of the image?
45. Suppose that Dinah wishes to use the photograph on a web page and wants the image to be 32 pixels wide. What scale factor should she use to reduce the image?
46. Dinah resizes the photograph so that it is 600 pixels high. What scale factor did she use?
47. **DESKTOP PUBLISHING** Grace is creating a template for her class newsletter. She has a photograph that is 10 centimeters by 12 centimeters, but the maximum space available for the photograph is 6 centimeters by 8 centimeters. She wants the photograph to be as large as possible on the page. When she uses a scanner to save the photograph, at what percent of the original photograph's size should she save the image file?

For Exercises 48–50, use quadrilateral $ABCD$.

48. Find the perimeter of quadrilateral $ABCD$.
49. Graph the image of quadrilateral $ABCD$ after a dilation centered at the origin with scale factor -2 .
50. Find the perimeter of quadrilateral $A'B'C'D'$ and compare it to the perimeter of quadrilateral $ABCD$.
51. Triangle TUV has vertices $T(6, -5)$, $U(3, -8)$, and $V(-1, -2)$. Find the coordinates of the final image of triangle TUV after a reflection in the x -axis, a translation with $(x, y) \rightarrow (x + 4, y - 1)$, and a dilation centered at the origin with a scale factor of $\frac{1}{3}$. Sketch the preimage and the image.



52. **CRITICAL THINKING** In order to perform a dilation not centered at the origin, you must first translate all of the points so the center is the origin, dilate the figure, and translate the points back. Consider a triangle with vertices $G(3, 5)$, $H(7, -4)$, and $I(-1, 0)$. State the coordinates of the vertices of the image after a dilation centered at $(3, 5)$ with a scale factor of 2.
53. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How do you use dilations when you use a computer?

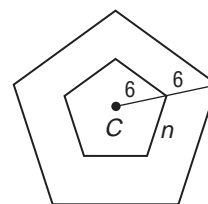
Include the following in your answer:

- how a “cut and paste” in word processing may be an example of a dilation, and
- other examples of dilations when using a computer.

Standardized Test Practice

54. The figure shows two regular pentagons. Find the perimeter of the larger pentagon.

- (A) $5n$ (B) $10n$
(C) $15n$ (D) $60n$



55. **ALGEBRA** What is the slope of a line perpendicular to the line given by the equation $3x + 5y = 12$?

- (A) $\frac{5}{3}$ (B) $\frac{3}{5}$ (C) $-\frac{3}{5}$ (D) $-\frac{5}{3}$

Maintain Your Skills

Mixed Review Determine whether a semi-regular tessellation can be created from each figure. Assume that each figure is regular and has a side length of 1 unit. (Lesson 9-4)

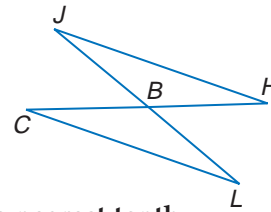
56. a triangle and a pentagon
 57. an octagon and a hexagon
 58. a square and a triangle
 59. a hexagon and a dodecagon

COORDINATE GEOMETRY Draw the rotation image of each figure 90° in the given direction about the center point and label the coordinates. (Lesson 9-3)

60. $\triangle ABC$ with $A(7, -1)$, $B(5, 0)$, and $C(1, 6)$ counterclockwise about $P(-1, 4)$
 61. $\square DEFG$ with $D(-4, -2)$, $E(-3, 3)$, $F(3, 1)$, and $G(2, -4)$ clockwise about $P(-4, -6)$

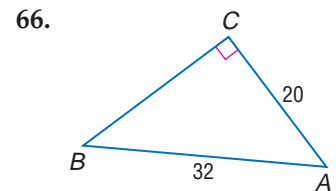
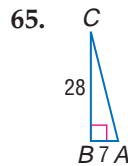
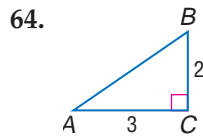
62. **CONSTRUCTION** The Vanamans are building an addition to their house. Ms. Vanaman is cutting an opening for a new window. If she measures to see that the opposite sides are the same length and that the diagonal measures are the same, can Ms. Vanaman be sure that the window opening is rectangular? Explain. (Lesson 8-4)

63. **Given:** $\angle J \cong \angle L$
 B is the midpoint of \overline{JL} .
Prove: $\triangle JHB \cong \triangle LCB$
 (Lesson 4-4)



Getting Ready for the Next Lesson

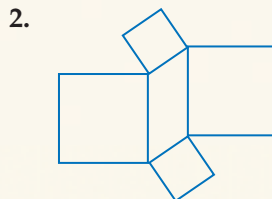
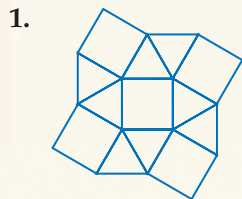
PREREQUISITE SKILL Find $m\angle A$ to the nearest tenth. (To review finding angles using inverses of trigonometric ratios, see Lesson 7-3.)



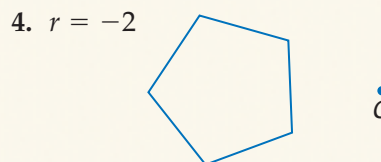
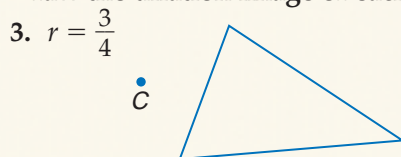
Practice Quiz 2

Lessons 9-4 and 9-5

Determine whether each pattern is a tessellation. If so, describe it as *uniform*, *regular*, *semi-regular*, or *not uniform*. (Lesson 9-4)



Draw the dilation image of each figure with center C and given scale factor. (Lesson 9-5)



5. Triangle ABC has vertices $A(10, 2)$, $B(1, 6)$, and $C(-4, 4)$. Find the image of $\triangle ABC$ after a dilation centered at the origin with scale factor of $-\frac{1}{2}$. Sketch the preimage and the image. (Lesson 9-5)



9-6 Vectors

What You'll Learn

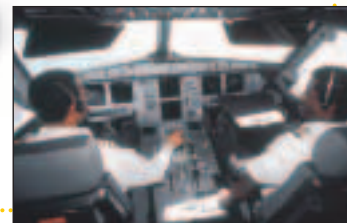
- Find magnitudes and directions of vectors.
- Perform translations with vectors.

Vocabulary

- vector
- magnitude
- direction
- standard position
- component form
- equal vectors
- parallel vectors
- resultant
- scalar
- scalar multiplication

How do vectors help a pilot plan a flight?

Commercial pilots must submit flight plans prior to departure. These flight plans take into account the speed and direction of the plane as well as the speed and direction of the wind.

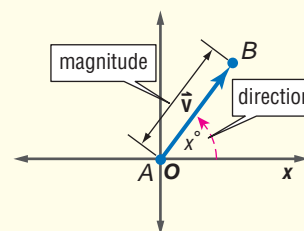


MAGNITUDE AND DIRECTION The speed and direction of a plane and the wind can be represented by vectors.

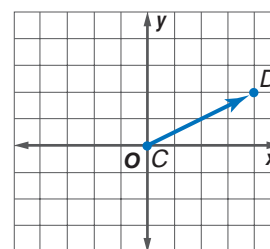
Key Concept

Vectors

- **Words** A **vector** is a quantity that has both **magnitude**, or length, and **direction**, and is represented by a directed segment.
- **Symbols** \vec{v}
 \overline{AB} , where A is the initial point and B is the endpoint



A vector in **standard position** has its initial point at the origin. In the diagram, \overline{CD} is in standard position and can be represented by the ordered pair $\langle 4, 2 \rangle$.



A vector can also be drawn anywhere in the coordinate plane. To write such a vector as an ordered pair, find the change in the x values and the change in y values, $\langle \text{change in } x, \text{change in } y \rangle$, from the tip to the tail of the directed segment. The ordered pair representation of a vector is called the **component form** of the vector.

Study Tip

Common Misconception

The notation for a vector from C to D , \overline{CD} , is similar to the notation for a ray from C to D , \overrightarrow{CD} . Be sure to use the correct arrow above the letters when writing each.

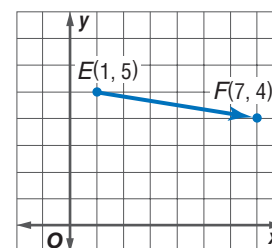
Example 1 Write Vectors in Component Form

Write the component form of \overline{EF} .

Find the change in x -values and the corresponding change in y -values.

$$\begin{aligned} \overline{EF} &= \langle x_2 - x_1, y_2 - y_1 \rangle && \text{Component form of vector} \\ &= \langle 7 - 1, 4 - 5 \rangle && x_1 = 1, y_1 = 5, x_2 = 7, y_2 = 4 \\ &= \langle 6, -1 \rangle && \text{Simplify.} \end{aligned}$$

Because the magnitude and direction of a vector are not changed by translation, the vector $\langle 6, -1 \rangle$ represents the same vector as \overline{EF} .



The Distance Formula can be used to find the magnitude of a vector. The symbol for the magnitude of \overline{AB} is $|\overline{AB}|$. The *direction* of a vector is the measure of the angle that the vector forms with the positive x -axis or any other horizontal line. You can use the trigonometric ratios to find the direction of a vector.

Example 2 Magnitude and Direction of a Vector

Find the magnitude and direction of \overline{PQ} for $P(3, 8)$ and $Q(-4, 2)$.

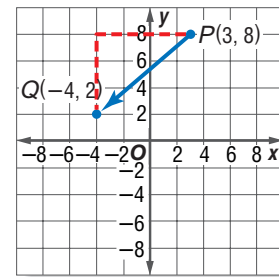
Find the magnitude.

$$\begin{aligned} |\overline{PQ}| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\ &= \sqrt{(-4 - 3)^2 + (2 - 8)^2} && x_1 = 3, y_1 = 8, x_2 = -4, y_2 = 2 \\ &= \sqrt{85} && \text{Simplify.} \\ &\approx 9.2 && \text{Use a calculator.} \end{aligned}$$

Graph \overline{PQ} to determine how to find the direction. Draw a right triangle that has \overline{PQ} as its hypotenuse and an acute angle at P .

$$\begin{aligned} \tan P &= \frac{y_2 - y_1}{x_2 - x_1} && \tan = \frac{\text{length of opposite side}}{\text{length of adjacent side}} \\ &= \frac{2 - 8}{-4 - 3} && \text{Substitution} \\ &= \frac{6}{7} && \text{Simplify.} \end{aligned}$$

$$\begin{aligned} m\angle P &= \tan^{-1} \frac{6}{7} \\ &\approx 40.6 && \text{Use a calculator.} \end{aligned}$$



A vector in standard position that is equal to \overline{PQ} forms a 40.6° angle with the negative x -axis in the third quadrant. So it forms a $180 + 40.6$ or 220.6° angle with the positive x -axis.

Thus, \overline{PQ} has a magnitude of about 9.2 units and a direction of about 220.6° .

In Example 1, we stated that a vector in standard position with magnitude of 9.2 units and a direction of 220.6° was equal to \overline{PQ} . This leads to a definition of equal vectors.

Study Tip

Using Slope

Even though slope is not associated with vectors, you can use the concept of slope to determine if vectors are parallel before actually comparing their directions.

Key Concept

Equal Vectors Two vectors are equal if and only if they have the same magnitude and direction.

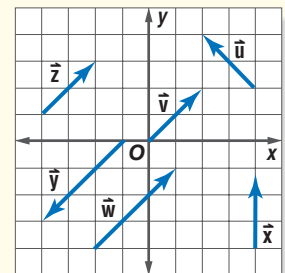
Example $\vec{v} = \vec{z}$

Nonexample $\vec{v} \neq \vec{u}, \vec{w} \neq \vec{y}$

Parallel Vectors Two vectors are parallel if and only if they have the same or opposite direction.

Example $\vec{v} \parallel \vec{w} \parallel \vec{y} \parallel \vec{z}$

Nonexample $\vec{v} \not\parallel \vec{x}$



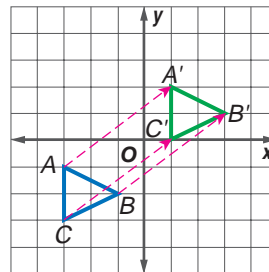
TRANSLATIONS WITH VECTORS

Vectors can be used to describe translations.

Example 3 Translations with Vectors

Graph the image of $\triangle ABC$ with vertices $A(-3, -1)$, $B(-1, -2)$, and $C(-3, -3)$ under the translation $\vec{v} = \langle 4, 3 \rangle$.

First, graph $\triangle ABC$. Next, translate each vertex by \vec{v} , 4 units right and 3 units up. Connect the vertices to form $\triangle A'B'C'$.



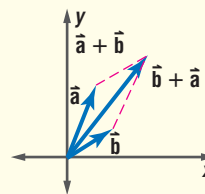
Vectors can be combined to perform a composition of translations by adding the vectors. To add vectors, add their corresponding components. The sum of two vectors is called the **resultant**.

Key Concept

Vector Addition

- **Words** To add two vectors, add the corresponding components.
- **Symbols** If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, and $\vec{b} + \vec{a} = \langle b_1 + a_1, b_2 + a_2 \rangle$.

• Model



Example 4 Add Vectors

Graph the image of $\square QRST$ with vertices $Q(-4, 4)$, $R(-1, 4)$, $S(-2, 2)$, and $T(-5, 2)$ under the translation $\vec{m} = \langle 5, -1 \rangle$ and $\vec{n} = \langle -2, -6 \rangle$.

Graph $\square QRST$.

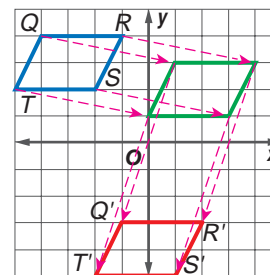
Method 1 Translate two times.

Translate $\square QRST$ by \vec{m} . Then translate $\square QRST$ by \vec{n} .

Translate each vertex 5 units right and 1 unit down.

Label the image $\square Q'R'S'T'$.

Then translate each vertex of $\square Q'R'S'T'$ 2 units left and 6 units down.



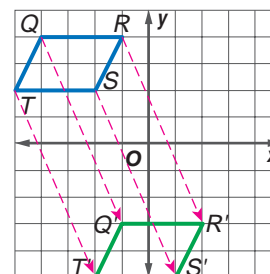
Method 2 Find the resultant, and then translate.

Add \vec{m} and \vec{n} .

$$\begin{aligned}\vec{m} + \vec{n} &= \langle 5 - 2, -1 - 6 \rangle \\ &= \langle 3, -7 \rangle\end{aligned}$$

Translate each vertex 3 units right and 7 units down.

Notice that the vertices for the image are the same for either method.





Geometry Activity

Comparing Magnitude and Components of Vectors

Model and Analyze

- Draw \vec{a} in standard position.
- Draw \vec{b} in standard position with the same direction as \vec{a} , but with a magnitude twice the magnitude of \vec{a} .

1. Write \vec{a} and \vec{b} in component form.
2. What do you notice about the components of \vec{a} and \vec{b} ?
3. Draw \vec{b} so that its magnitude is three times that of \vec{a} . How do the components of \vec{a} and \vec{b} compare?

Make a Conjecture

4. Describe the vector magnitude and direction of a vector $\langle x, y \rangle$ after the components are multiplied by n .

In the Geometry Activity, you found that a vector can be multiplied by a positive constant, called a **scalar**, that will change the magnitude of the vector, but not affect its direction. Multiplying a vector by a positive scalar is called **scalar multiplication**.

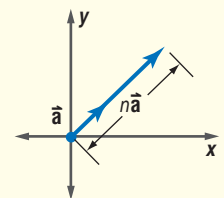
Key Concept

Scalar Multiplication

• **Words** To multiply a vector by a scalar multiply each component by the scalar.

• **Symbols** If $\vec{a} = \langle a_1, a_2 \rangle$ has a magnitude $|\vec{a}|$ and direction d , then $n\vec{a} = n\langle a_1, a_2 \rangle = \langle na_1, na_2 \rangle$, where n is a positive real number, the magnitude is $|n\vec{a}|$, and its direction is d .

• **Model**



Example 5 Solve Problems Using Vectors

• **AVIATION** Refer to the application at the beginning of the lesson.

- a. Suppose a pilot begins a flight along a path due north flying at 250 miles per hour. If the wind is blowing due east at 20 miles per hour, what is the resultant velocity and direction of the plane?

The initial path of the plane is due north, so a vector representing the path lies on the positive y -axis 250 units long. The wind is blowing due east, so a vector representing the wind will be parallel to the positive x -axis 20 units long. The resultant path can be represented by a vector from the initial point of the vector representing the plane to the terminal point of the vector representing the wind.

Use the Pythagorean Theorem.

$$c^2 = a^2 + b^2 \quad \text{Pythagorean Theorem}$$

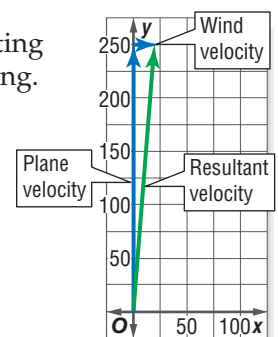
$$c^2 = 250^2 + 20^2 \quad a = 250, b = 20$$

$$c^2 = 62,900 \quad \text{Simplify.}$$

$$c = \sqrt{62,900} \quad \text{Take the square root of each side.}$$

$$c \approx 250.8$$

The resultant speed of the plane is about 250.8 miles per hour.



More About . . .



Aviation

A tailwind will allow a plane to arrive faster than anticipated without the tailwind. A headwind will cause the plane to take more time to travel than without the headwind.

(continued on the next page)

Study Tip

Reading Math

The Greek letter theta, θ , is used to represent the unknown measure of an angle.

Use the tangent ratio to find the direction of the plane.

$$\tan \theta = \frac{20}{250} \quad \text{side opposite} = 20, \text{ side adjacent} = 250$$

$$\theta = \tan^{-1} \frac{20}{250} \quad \text{Solve for } \theta.$$

$$\theta \approx 4.6 \quad \text{Use a calculator.}$$

The resultant direction of the plane is about 4.6° east of due north.

Therefore, the resultant vector is 250.8 miles per hour at 4.6° east of due north.

- b. If the wind velocity doubles, what is the resultant path and velocity of the plane?**

Use scalar multiplication to find the magnitude of the vector for wind velocity.

$$n|\vec{a}| = 2|20| \quad \text{Magnitude of } n\vec{a}; n = 2, |\vec{a}| = 20$$

$$= 2(20) \text{ or } 40 \quad \text{Simplify.}$$

Next, use the Pythagorean Theorem to find the magnitude of the resultant vector.

$$c^2 = a^2 + b^2 \quad \text{Pythagorean Theorem}$$

$$c^2 = 250^2 + 40^2 \quad a = 250, b = 40$$

$$c^2 = 64,100 \quad \text{Simplify.}$$

$$c = \sqrt{64,100} \quad \text{Take the square root of each side.}$$

$$c \approx 253.2$$

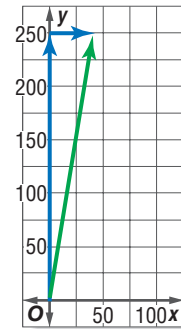
Then, use the tangent ratio to find the direction of the plane.

$$\tan \theta = \frac{40}{250} \quad \text{side opposite} = 40, \text{ side adjacent} = 250$$

$$\theta = \tan^{-1} \frac{40}{250} \quad \text{Solve for } \theta.$$

$$\theta \approx 9.1 \quad \text{Use a calculator.}$$

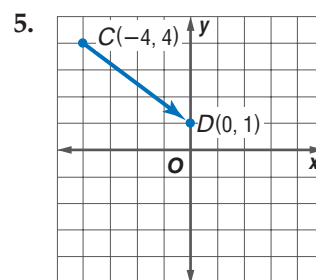
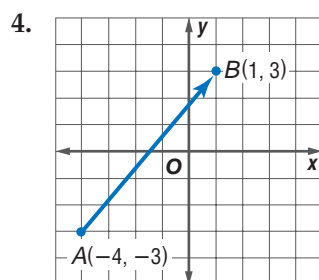
If the wind velocity doubles, the plane flies along a path approximately 9.1° east of due north at about 253.2 miles per hour.



Check for Understanding

- Concept Check**
- OPEN ENDED** Draw a pair of vectors on a coordinate plane. Label each vector in component form and then find their sum.
 - Compare and contrast** equal vectors and parallel vectors.
 - Discuss** the similarity of using vectors to translate a figure and using an ordered pair.

Guided Practice Write the component form of each vector.



Find the magnitude and direction of \overline{MN} for the given coordinates. Round to the nearest tenth.

31. $M(-3, 3), N(-9, 9)$ 32. $M(8, 1), N(2, 5)$ 33. $M(0, 2), N(-12, -2)$
 34. $M(-1, 7), N(6, -8)$ 35. $M(-1, 10), N(1, -12)$ 36. $M(-4, 0), N(-6, -4)$

Graph the image of each figure under a translation by the given vector.

37. $\triangle ABC$ with vertices $A(3, 6), B(3, -7), C(-6, 1)$; $\vec{a} = \langle 0, -6 \rangle$
 38. $\triangle DEF$ with vertices $D(-12, 6), E(7, 6), F(7, -3)$; $\vec{b} = \langle -3, -9 \rangle$
 39. square $GHIJ$ with vertices $G(-1, 0), H(-6, -3), I(-9, 2), J(-4, 5)$; $\vec{c} = \langle 3, -8 \rangle$
 40. quadrilateral $KLMN$ with vertices $K(0, 8), L(4, 6), M(3, -3), N(-4, 8)$; $\vec{x} = \langle -10, 2 \rangle$
 41. pentagon $OPQRS$ with vertices $O(5, 3), P(5, -3), Q(0, -4), R(-5, 0), S(0, 4)$; $\vec{y} = \langle -5, 11 \rangle$
 42. hexagon $TUVWXYZ$ with vertices $T(4, -2), U(3, 3), V(6, 4), W(9, 3), X(8, -2), Y(6, -5)$; $\vec{z} = \langle -18, 12 \rangle$

Graph the image of each figure under a translation by the given vectors.

43. $\square ABCD$ with vertices $A(-1, -6), B(4, -8), C(-3, -11), D(-8, -9)$; $\vec{p} = \langle 11, 6 \rangle, \vec{q} = \langle -9, -3 \rangle$
 44. $\triangle XYZ$ with vertices $X(3, -5), Y(9, 4), Z(12, -2)$; $\vec{p} = \langle 2, 2 \rangle, \vec{q} = \langle -4, -7 \rangle$
 45. quadrilateral $EFGH$ with vertices $E(-7, -2), F(-3, 8), G(4, 15), H(5, -1)$; $\vec{p} = \langle -6, 10 \rangle, \vec{q} = \langle 1, -8 \rangle$
 46. pentagon $STUVW$ with vertices $S(1, 4), T(3, 8), U(6, 8), V(6, 6), W(4, 4)$; $\vec{p} = \langle -4, 5 \rangle, \vec{q} = \langle 12, 11 \rangle$

Find the magnitude and direction of each resultant for the given vectors.

47. $\vec{a} = \langle 5, 0 \rangle, \vec{b} = \langle 0, 12 \rangle$ 48. $\vec{c} = \langle 0, -8 \rangle, \vec{d} = \langle -8, 0 \rangle$
 49. $\vec{e} = \langle -4, 0 \rangle, \vec{f} = \langle 7, -4 \rangle$ 50. $\vec{u} = \langle 12, 6 \rangle, \vec{v} = \langle 0, 6 \rangle$
 51. $\vec{w} = \langle 5, 6 \rangle, \vec{x} = \langle -1, -4 \rangle$ 52. $\vec{y} = \langle 9, -10 \rangle, \vec{z} = \langle -10, -2 \rangle$

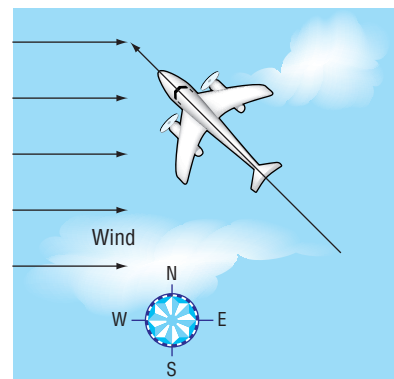
53. **SHIPPING** A freighter has to go around an oil spill in the Pacific Ocean. The captain sails due east for 35 miles. Then he turns the ship and heads due south for 28 miles. What is the distance and direction of the ship from its original point of course correction?

54. **RIVERS** Suppose a section of the New River in West Virginia has a current of 2 miles per hour. If a swimmer can swim at a rate of 4.5 miles per hour, how does the current in the New River affect the speed and direction of the swimmer as she tries to swim directly across the river?

AVIATION For Exercises 55–57, use the following information.

A jet is flying northwest, and its velocity is represented by $\langle -450, 450 \rangle$ miles per hour. The wind is from the west, and its velocity is represented by $\langle 100, 0 \rangle$ miles per hour.

55. Find the resultant vector for the jet in component form.
 56. Find the magnitude of the resultant.
 57. Find the direction of the resultant.



More About . . .



Rivers

The Congo River is one of the fastest rivers in the world. It has no dry season, because it has tributaries both north and south of the Equator. The river flows so quickly that it doesn't form a delta where it ends in the Atlantic like most rivers do when they enter an ocean.

Source: Compton's Encyclopedia

58. **CRITICAL THINKING** If two vectors have opposite directions but the same magnitude, the resultant is $\langle 0, 0 \rangle$ when they are added. Find three vectors of equal magnitude, each with its tail at the origin, the sum of which is $\langle 0, 0 \rangle$.
59. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How do vectors help a pilot plan a flight?

Include the following in your answer:

- an explanation of how a wind from the west affects the overall velocity of a plane traveling east, and
- an explanation as to why planes traveling from Hawaii to the continental U.S. take less time than planes traveling from the continental U.S. to Hawaii.



60. If $\vec{q} = \langle 5, 10 \rangle$ and $\vec{r} = \langle 3, 5 \rangle$, find the magnitude for the sum of these two vectors.
- (A) 23 (B) 17 (C) 7 (D) $\sqrt{29}$
61. **ALGEBRA** If $5^b = 125$, then find $4^b \times 3$.
- (A) 48 (B) 64 (C) 144 (D) 192

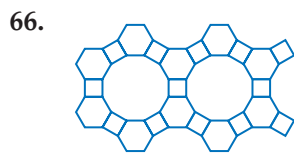
Maintain Your Skills

Mixed Review

Find the measure of the dilation image $\overline{A'B'}$ or the preimage of \overline{AB} using the given scale factor. (Lesson 9-5)

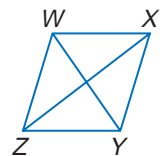
62. $AB = 8, r = 2$ 63. $AB = 12, r = \frac{1}{2}$
64. $A'B' = 15, r = 3$ 65. $A'B' = 12, r = \frac{1}{4}$

Determine whether each pattern is a tessellation. If so, describe it as *uniform, not uniform, regular, or semi-regular*. (Lesson 9-4)



ALGEBRA Use rhombus $WXYZ$ with $m\angle XYZ = 5m\angle WZY$ and $YZ = 12$. (Lesson 8-5)

68. Find $m\angle XYZ$. 69. Find WX .
70. Find $m\angle XZY$. 71. Find $m\angle WXY$.



72. Each side of a rhombus is 30 centimeters long. One diagonal makes a 25° angle with a side. What is the length of each diagonal to the nearest tenth? (Lesson 7-4)

Getting Ready for the Next Lesson

PREREQUISITE SKILL Perform the indicated operation.

(To review operations with matrices, see pages 752–753.)

73. $\begin{bmatrix} -5 & 5 \\ -3 & -2 \end{bmatrix} + \begin{bmatrix} 1 & -8 \\ -7 & 6 \end{bmatrix}$ 74. $\begin{bmatrix} -2 & 2 & -2 \\ -7 & -2 & -5 \end{bmatrix} + \begin{bmatrix} -8 & -8 & -8 \\ 1 & 1 & 1 \end{bmatrix}$
75. $3 \begin{bmatrix} -9 & -5 & -1 \\ 9 & 1 & 5 \end{bmatrix}$ 76. $\frac{1}{2} \begin{bmatrix} -4 & -5 & 0 & 2 \\ 4 & 4 & 6 & 0 \end{bmatrix}$
77. $\begin{bmatrix} -4 & -4 \\ 2 & 2 \end{bmatrix} + 2 \begin{bmatrix} 8 & 4 \\ -3 & -7 \end{bmatrix}$ 78. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -2 & -4 \end{bmatrix}$



What You'll Learn

- Use matrices to determine the coordinates of translations and dilations.
- Use matrices to determine the coordinates of reflections and rotations.

Vocabulary

- column matrix
- vertex matrix
- translation matrix
- reflection matrix
- rotation matrix

How can matrices be used to make movies?

Many movie directors use computers to create special effects that cannot be easily created in real life. A special effect is often a simple image that is enhanced using transformations. Complex images can be broken down into simple polygons, which are moved and resized using matrices to define new vertices for the polygons.



TRANSLATIONS AND DILATIONS In Lesson 9-6, you learned that a vector can be represented by the ordered pair $\langle x, y \rangle$. A vector can also be represented by a **column matrix** $\begin{bmatrix} x \\ y \end{bmatrix}$. Likewise, polygons can be represented by placing all of the column matrices of the coordinates of the vertices into one matrix, called a **vertex matrix**.

Triangle PQR with vertices $P(3, 5)$, $Q(1, -2)$, and $R(-4, 4)$ can be represented by the vertex matrix at the right.

$$\Delta PQR = \begin{matrix} & P & Q & R \\ \begin{bmatrix} 3 & 1 & -4 \\ 5 & -2 & 4 \end{bmatrix} & \leftarrow & \text{x-coordinates} \\ & & & \leftarrow & \text{y-coordinates} \end{matrix}$$

Like vectors, matrices can be used to perform translations. You can use matrix addition and a **translation matrix** to find the coordinates of a translated figure.

Study Tip

Translation Matrix

A translation matrix contains the same number of rows and columns as the vertex matrix of a figure.

Example 1 Translate a Figure

Use a matrix to find the coordinates of the vertices of the image of $\square ABCD$ with $A(3, 2)$, $B(1, -3)$, $C(-3, -1)$, and $D(-1, 4)$ under the translation $(x, y) \rightarrow (x + 5, y - 3)$.

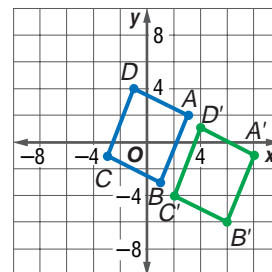
Write the vertex matrix for $\square ABCD$.

$$\begin{bmatrix} 3 & 1 & -3 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix}$$

To translate the figure 5 units to the right, add 5 to each x -coordinate. To translate the figure 3 units down, add -3 to each y -coordinate. This can be done by adding the translation matrix $\begin{bmatrix} 5 & 5 & 5 & 5 \\ -3 & -3 & -3 & -3 \end{bmatrix}$ to the vertex matrix of $\square ABCD$.

$$\begin{matrix} \text{Vertex Matrix} & & \text{Translation} & & \text{Vertex Matrix} \\ \text{of } \square ABCD & & \text{Matrix} & & \text{of } \square A'B'C'D' \\ \begin{bmatrix} 3 & 1 & -3 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix} & + & \begin{bmatrix} 5 & 5 & 5 & 5 \\ -3 & -3 & -3 & -3 \end{bmatrix} & = & \begin{bmatrix} 8 & 6 & 2 & 4 \\ -1 & -6 & -4 & 1 \end{bmatrix} \end{matrix}$$

The coordinates of $\square A'B'C'D'$ are $A'(8, -1)$, $B'(6, -6)$, $C'(2, -4)$, and $D'(4, 1)$.



Scalars can be used with matrices to perform dilations.

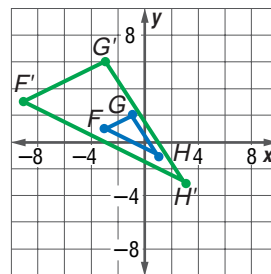
Example 2 Dilate a Figure

Triangle FGH has vertices $F(-3, 1)$, $G(-1, 2)$, and $H(1, -1)$. Use scalar multiplication to dilate $\triangle FGH$ centered at the origin so that its perimeter is 3 times the original perimeter.

If the perimeter of a figure is 3 times the original perimeter, then the lengths of the sides of the figure will be 3 times the measures of the original lengths. Multiply the vertex matrix by a scale factor of 3.

$$3 \begin{bmatrix} -3 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -9 & -3 & 3 \\ 3 & 6 & -3 \end{bmatrix}$$

The coordinates of the vertices of $\triangle F'G'H'$ are $F'(-9, 3)$, $G'(-3, 6)$, and $H'(3, -3)$.



REFLECTIONS AND ROTATIONS

A **reflection matrix** can be used to multiply the vertex matrix of a figure to find the coordinates of the image. The matrices used for four common reflections are shown below.

Concept Summary

Reflection Matrices

For a reflection in the:	x -axis	y -axis	origin	line $y = x$
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
The product of the reflection matrix and the vertex matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ would be:	$\begin{bmatrix} a & b \\ -c & -d \end{bmatrix}$	$\begin{bmatrix} -a & -b \\ c & d \end{bmatrix}$	$\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$	$\begin{bmatrix} c & d \\ a & b \end{bmatrix}$

Study Tip

Reflection Matrices

The matrices used for reflections and rotations always have two rows and two columns no matter what the number of columns in the vertex matrix.

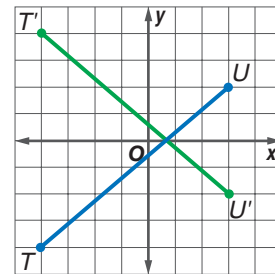
Example 3 Reflections

Use a matrix to find the coordinates of the vertices of the image of \overline{TU} with $T(-4, -4)$ and $U(3, 2)$ after a reflection in the x -axis.

Write the ordered pairs as a vertex matrix. Then multiply the vertex matrix by the reflection matrix for the x -axis.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 4 & -2 \end{bmatrix}$$

The coordinates of the vertices of $\overline{T'U'}$ are $T'(-4, 4)$ and $U'(3, -2)$. Graph \overline{TU} and $\overline{T'U'}$.



Matrices can also be used to determine the vertices of a figure's image by rotation using a **rotation matrix**. Commonly used rotation matrices are summarized on the next page.



Concept Summary

Rotation Matrices

For a counterclockwise rotation about the origin of:	90°	180°	270°
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
The product of the rotation matrix and the vertex matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ would be:	$\begin{bmatrix} -c & -d \\ a & b \end{bmatrix}$	$\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$	$\begin{bmatrix} c & d \\ -a & -b \end{bmatrix}$

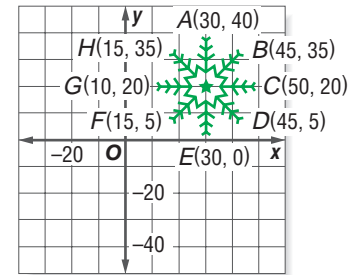
Study Tip

Graphing Calculator

It may be helpful to store the reflection and rotation matrices in your calculator to avoid reentering them.

Example 4 Use Rotations

COMPUTERS A software designer is creating a screen saver by transforming various patterns like the one at the right.



- a. Write a vertex matrix for the figure.

One possible vertex matrix is

$$\begin{bmatrix} 30 & 45 & 50 & 45 & 30 & 15 & 10 & 15 \\ 40 & 35 & 20 & 5 & 0 & 5 & 20 & 35 \end{bmatrix}$$

- b. Use a matrix to find the coordinates of the image under a 90° counterclockwise rotation about the origin.

Enter the rotation matrix in your calculator as matrix A and enter the vertex matrix as matrix B. Then multiply.

KEYSTROKES: $\boxed{2\text{nd}}$ $\boxed{[\text{MATRX}]}$ $\boxed{1}$ $\boxed{2\text{nd}}$ $\boxed{[\text{MATRX}]}$ $\boxed{2}$ $\boxed{\text{ENTER}}$

$$AB = \begin{bmatrix} -40 & -35 & -20 & -5 & 0 & -5 & -20 & -35 \\ 30 & 45 & 50 & 45 & 30 & 15 & 10 & 15 \end{bmatrix}$$

The coordinates of the vertices of the figure are $A'(-40, 30)$, $B'(-35, 45)$, $C'(-20, 50)$, $D'(-5, 45)$, $E'(0, 30)$, $F'(-5, 15)$, $G'(-20, 10)$, and $H'(-35, 15)$.

- c. What are the coordinates of the image if the pattern is enlarged to twice its size before a 180° rotation about the origin?

Enter the rotation matrix as matrix C. A dilation is performed before the rotation. Multiply the matrices by 2.

KEYSTROKES: $\boxed{2}$ $\boxed{2\text{nd}}$ $\boxed{[\text{MATRX}]}$ $\boxed{3}$ $\boxed{2\text{nd}}$ $\boxed{[\text{MATRX}]}$ $\boxed{2}$ $\boxed{\text{ENTER}}$

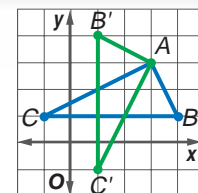
$$2AC = \begin{bmatrix} -60 & -90 & -100 & -90 & -60 & -30 & -20 & -30 \\ -80 & -70 & -40 & -10 & 0 & -10 & -40 & -70 \end{bmatrix}$$

The coordinates of the vertices of the figure are $A'(-60, -80)$, $B'(-90, -70)$, $C'(-100, -40)$, $D'(-90, -10)$, $E'(-60, 0)$, $F'(-30, -10)$, $G'(-20, -40)$, and $H'(-30, -70)$.

Check for Understanding

Concept Check

- Write the reflection matrix for $\triangle ABC$ and its image $\triangle A'B'C'$ at the right.
- Discuss the similarities of using coordinates, vectors, and matrices to translate a figure.
- OPEN ENDED** Graph any $\square PQRS$ on a coordinate grid. Then write a translation matrix that moves $\square PQRS$ down and left on the grid.



Guided Practice

Use a matrix to find the coordinates of the vertices of the image of each figure under the given translations.

- $\triangle ABC$ with $A(5, 4)$, $B(3, -1)$, and $C(0, 2)$; $(x, y) \rightarrow (x - 2, y - 1)$
- rectangle $DEFG$ with $D(-1, 3)$, $E(5, 3)$, $F(3, 0)$, and $G(-3, 0)$; $(x, y) \rightarrow (x, y + 6)$

Use scalar multiplication to find the coordinates of the vertices of each figure for a dilation centered at the origin with the given scale factor.

- $\triangle XYZ$ with $X(3, 4)$, $Y(6, 10)$, and $Z(-3, 5)$; $r = 2$
- $\square ABCD$ with $A(1, 2)$, $B(3, 3)$, $C(3, 5)$, and $D(1, 4)$; $r = -\frac{1}{4}$

Use a matrix to find the coordinates of the vertices of the image of each figure under the given reflection.

- \overline{EF} with $E(-2, 4)$ and $F(5, 1)$; x -axis
- $\square HIJK$ with $H(-5, 4)$, $I(-1, -1)$, $J(-3, -6)$, and $K(-7, -3)$; y -axis

Use a matrix to find the coordinates of the vertices of the image of each figure under the given rotation.

- \overline{LM} with $L(-2, 1)$ and $M(3, 5)$; 90° counterclockwise
- $\triangle PQR$ with $P(6, 3)$, $Q(6, 7)$, and $R(2, 7)$; 270° counterclockwise
- Use matrices to find the coordinates of the image of quadrilateral $STUV$ with $S(-4, 1)$, $T(-2, 2)$, $U(0, 1)$, and $V(-2, -2)$ after a dilation by a scale factor of 2 and a rotation 90° counterclockwise about the origin.

Application

LANDSCAPING For Exercises 13 and 14, use the following information.

A garden design is drawn on a coordinate grid. The original plan shows a rose bed with vertices at $(3, -1)$, $(7, -3)$, $(5, -7)$, and $(1, -5)$. Changes to the plan require that the rose bed's perimeter be half the original perimeter with respect to the origin, while the shape remains the same.

- What are the new coordinates for the vertices of the rose bed?
- If the center of the rose bed was originally located at $(4, -4)$, what will be the coordinates of the center after the changes have been made?

Practice and Apply

Homework Help

For Exercises	See Examples
15–18, 28, 31, 40, 41	1
19–22, 27, 32, 39, 42	2
23–26, 30, 33, 39, 41, 29, 34,	3
35–38, 40, 42	4

Extra Practice
See page 773.

Use a matrix to find the coordinates of the vertices of the image of each figure under the given translations.

- \overline{EF} with $E(-4, 1)$, and $F(-1, 3)$; $(x, y) \rightarrow (x - 2, y + 5)$
- $\triangle JKL$ with $J(-3, 5)$, $K(4, 8)$, and $L(7, 5)$; $(x, y) \rightarrow (x - 3, y - 4)$
- $\square MNOP$ with $M(-2, 7)$, $N(2, 9)$, $O(2, 7)$, and $P(-2, 5)$; $(x, y) \rightarrow (x + 3, y - 6)$
- trapezoid $RSTU$ with $R(2, 3)$, $S(6, 2)$, $T(6, -1)$, and $U(-2, 1)$; $(x, y) \rightarrow (x - 6, y - 2)$

Use scalar multiplication to find the coordinates of the vertices of each figure for a dilation centered at the origin with the given scale factor.

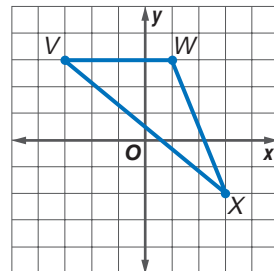
- $\triangle ABC$ with $A(6, 5)$, $B(4, 5)$, and $C(3, 7)$; $r = 2$
- $\triangle DEF$ with $D(-1, 4)$, $E(0, 1)$, and $F(2, 3)$; $r = -\frac{1}{3}$
- quadrilateral $GHIJ$ with $G(4, 2)$, $H(-4, 6)$, $I(-6, -8)$, and $J(6, -10)$; $r = -\frac{1}{2}$
- pentagon $KLMNO$ with $K(1, -2)$, $L(3, -1)$, $M(6, -1)$, $N(4, -3)$, and $O(3, -3)$; $r = 4$

Use a matrix to find the coordinates of the vertices of the image of each figure under the given reflection.

23. \overline{XY} with $X(2, 2)$, and $Y(4, -1)$; y -axis
24. $\triangle ABC$ with $A(5, -3)$, $B(0, -5)$, and $C(-1, -3)$; $y = x$
25. quadrilateral $DEFG$ with $D(-4, 5)$, $E(2, 6)$, $F(3, 1)$, and $G(-3, -4)$; x -axis
26. quadrilateral $HJKL$ with $H(9, -1)$, $I(2, -6)$, $J(-4, -3)$, and $K(-2, 4)$; y -axis

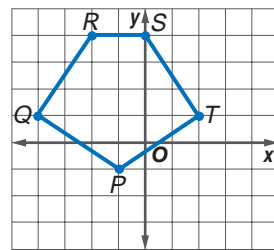
Find the coordinates of the image of $\triangle VWX$ under the stated transformation.

27. dilation by scale factor $\frac{2}{3}$
28. translation $(x, y) \rightarrow (x - 4, y - 1)$
29. rotation 90° counterclockwise about the origin
30. reflection in the line $y = x$



Find the coordinates of the image of polygon $PQRST$ under the stated transformation.

31. translation $(x, y) \rightarrow (x + 3, y - 2)$
32. dilation by scale factor -3
33. reflection in the y -axis
34. rotation 180° counterclockwise about the origin

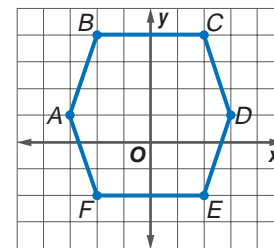


Use a matrix to find the coordinates of the vertices of the image of each figure under the given rotation.

35. \overline{MN} with $M(12, 1)$ and $N(-3, 10)$; 90° counterclockwise
36. $\triangle PQR$ with $P(5, 1)$, $Q(1, 2)$, and $R(1, -4)$; 180° counterclockwise
37. $\square STUV$ with $S(2, 1)$, $T(6, 1)$, $U(5, -3)$, and $V(1, -3)$; 90° counterclockwise
38. pentagon $ABCDE$ with $A(-1, 1)$, $B(6, 0)$, $C(4, -8)$, $D(-4, -10)$, and $E(-5, -3)$; 270° counterclockwise

Find the coordinates of the image under the stated transformations.

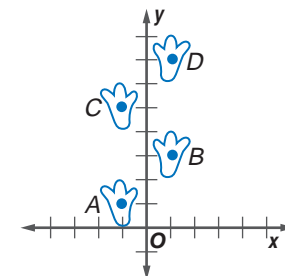
39. dilation by scale factor $\frac{1}{3}$, then a reflection in the x -axis
40. translation $(x, y) \rightarrow (x - 5, y + 2)$ then a rotation 90° counterclockwise about the origin
41. reflection in the line $y = x$ then the translation $(x, y) \rightarrow (x + 1, y + 4)$
42. rotation 180° counterclockwise about the origin, then a dilation by scale factor -2



PALEONTOLOGY For Exercises 43 and 44, use the following information.

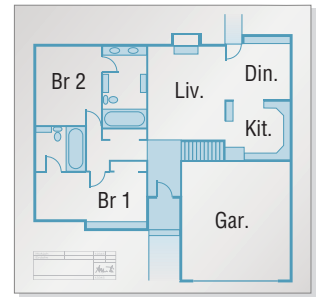
Paleontologists sometimes discover sets of fossilized dinosaur footprints like those shown at the right.

43. Describe the transformation combination shown.
44. Write two matrix operations that could be used to find the coordinates of point C.



CONSTRUCTION For Exercises 45 and 46, use the following information.

House builders often use one set of blueprints for many projects. By changing the orientation of a floor plan, a builder can make several different looking houses.



45. Write a transformation matrix that could be used to create a floor plan with the garage on the left.
46. If the current plan is of a house that faces south, write a transformation matrix that could be used to create a floor plan for a house that faces east.
47. **CRITICAL THINKING** Write a matrix to represent a reflection in the line $y = -x$.
48. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can matrices be used to make movies?

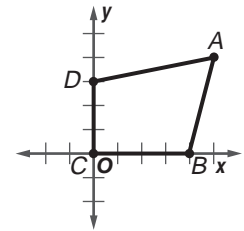
Include the following in your answer:

- an explanation of how transformations are used in movie production, and
- an everyday example of transformation that can be modeled using matrices.

Standardized Test Practice

- (A) (B) (C) (D)

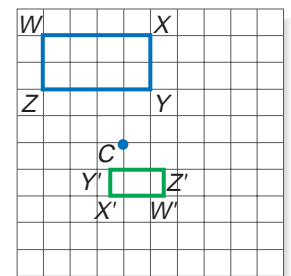
49. **SHORT RESPONSE** Quadrilateral $ABCD$ is rotated 90° clockwise about the origin. Write the transformation matrix.
50. **ALGEBRA** A video store stocks 2500 different movie titles. If 26% of the titles are action movies and 14% are comedies, how many are neither action movies nor comedies?
- (A) 1000 (B) 1500
(C) 1850 (D) 2150



Maintain Your Skills

Mixed Review Graph the image of each figure under a translation by the given vector. (Lesson 9-6)

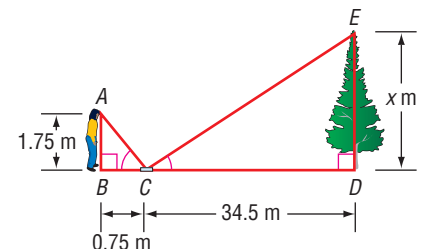
51. $\triangle ABC$ with $A(-6, 1)$, $B(4, 8)$, and $C(1, -4)$; $\vec{v} = \langle -1, -5 \rangle$
52. quadrilateral $DEFG$ with $D(3, -3)$, $E(1, 2)$, $F(8, -1)$, and $G(4, -6)$; $\vec{w} = \langle -7, 8 \rangle$
53. Determine the scale factor used for the dilation at the right, centered at C . Determine whether the dilation is an *enlargement*, *reduction*, or *congruence transformation*. (Lesson 9-5)



Find the measures of an exterior angle and an interior angle given the number of sides of a regular polygon. (Lesson 8-1)

54. 5 (B) 55. 6
56. 8 (C) 57. 10

58. **FORESTRY** To estimate the height of a tree, Lara sights the top of the tree in a mirror that is 34.5 meters from the tree. The mirror is on the ground and faces upward. Lara is standing 0.75 meter from the mirror, and the distance from her eyes to the ground is 1.75 meters. How tall is the tree? (Lesson 6-3)



Vocabulary and Concept Check

angle of rotation (p. 476)	line of reflection (p. 463)	scalar (p. 501)
center of rotation (p. 476)	line of symmetry (p. 466)	scalar multiplication (p. 501)
column matrix (p. 506)	magnitude (p. 498)	semi-regular tessellation (p. 484)
component form (p. 498)	parallel vectors (p. 499)	similarity transformation (p. 491)
composition of reflections (p. 471)	point of symmetry (p. 466)	standard position (p. 498)
dilation (p. 490)	reflection (p. 463)	tessellation (p. 483)
direct isometry (p. 481)	reflection matrix (p. 507)	transformation (p. 462)
direction (p. 498)	regular tessellation (p. 484)	translation (p. 470)
equal vectors (p. 499)	resultant (p. 500)	translation matrix (p. 506)
glide reflection (p. 474)	rotation (p. 476)	uniform (p. 484)
indirect isometry (p. 481)	rotation matrix (p. 507)	vector (p. 498)
isometry (p. 463)	rotational symmetry (p. 478)	vertex matrix (p. 506)

A complete list of postulates and theorems can be found on pages R1–R8.

Exercises State whether each sentence is *true* or *false*. If false, replace the underlined word or phrase to make a true sentence.

1. A dilation can change the distance between each point on the figure and the given line of symmetry.
2. A tessellation is uniform if the same combination of shapes and angles is present at every vertex.
3. Two vectors can be added easily if you know their magnitude.
4. Scalar multiplication affects only the direction of a vector.
5. In a rotation, the figure is turned about the point of symmetry.
6. A reflection is a transformation determined by a figure and a line.
7. A congruence transformation is the amount by which a figure is enlarged or reduced in a dilation.
8. A scalar multiple is the sum of two other vectors.

Lesson-by-Lesson Review

9-1 Reflections

See pages
463–469.

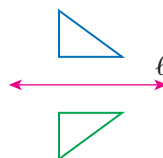
Concept Summary

- A reflection in a line or a point is a congruence transformation.
- The line of symmetry in a figure is a line where the figure could be folded in half so that the two halves match exactly.

Example

Copy the figure. Draw the image of the figure under a reflection in line ℓ .

The green triangle is the reflected image of the blue triangle.



Exercises Graph each figure and its image under the given reflection.

See Example 2 on page 464.

- triangle ABC with $A(2, 1)$, $B(5, 1)$, and $C(2, 3)$ in the x -axis
- parallelogram $WXYZ$ with $W(-4, 5)$, $X(-1, 5)$, $Y(-3, 3)$, and $Z(-6, 3)$ in the line $y = x$
- rectangle $EFGH$ with $E(-4, -2)$, $F(0, -2)$, $G(0, -4)$, and $H(-4, -4)$ in the line $x = 1$

9-2 Translations

See pages
470–475.

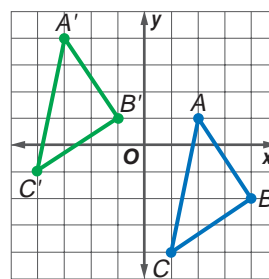
Concept Summary

- A translation moves all points of a figure the same distance in the same direction.
- A translation can be represented as a composition of reflections.

Example

COORDINATE GEOMETRY Triangle ABC has vertices $A(2, 1)$, $B(4, -2)$, and $C(1, -4)$. Graph $\triangle ABC$ and its image for the translation $(x, y) \rightarrow (x - 5, y + 3)$.

(x, y)	$(x - 5, y + 3)$
$(2, 1)$	$(-3, 4)$
$(4, -2)$	$(-1, 1)$
$(1, -4)$	$(-4, -1)$



This translation moved every point of the preimage 5 units left and 3 units up.

Exercises Graph each figure and the image under the given translation.

See Example 1 on page 470.

- quadrilateral $EFGH$ with $E(2, 2)$, $F(6, 2)$, $G(4, -2)$, $H(1, -1)$ under the translation $(x, y) \rightarrow (x - 4, y - 4)$
- \overline{ST} with endpoints $S(-3, -5)$, $T(-1, -1)$ under the translation $(x, y) \rightarrow (x + 2, y + 4)$
- $\triangle XYZ$ with $X(2, 5)$, $Y(1, 1)$, $Z(5, 1)$ under the translation $(x, y) \rightarrow (x + 1, y - 3)$

9-3 Rotations

See pages
476–482.

Concept Summary

- A rotation turns each point in a figure through the same angle about a fixed point.
- An object has rotational symmetry when you can rotate it less than 360° and the preimage and image are indistinguishable.

Example

Identify the order and magnitude of the rotational symmetry in the figure.

The figure has rotational symmetry of order 12 because there are 12 rotations of less than 360° (including 0°) that produce an image indistinguishable from the original.

The magnitude is $360^\circ \div 12$ or 30° .



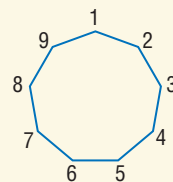
Exercises Draw the rotation image of each triangle by reflecting the triangles in the given lines. State the coordinates of the rotation image and the angle of rotation. See Example 2 on page 478.

15. $\triangle BCD$ with vertices $B(-3, 5)$, $C(-3, 3)$, and $D(-5, 3)$ reflected in the x -axis and then the y -axis
16. $\triangle FGH$ with vertices $F(0, 3)$, $G(-1, 0)$, $H(-4, 1)$ reflected in the line $y = x$ and then the line $y = -x$
17. $\triangle LMN$ with vertices $L(2, 2)$, $M(5, 3)$, $N(3, 6)$ reflected in the line $y = -x$ and then the x -axis

The figure at the right is a regular nonagon.

See Exercise 3 on page 478.

18. Identify the order and magnitude of the symmetry.
19. What is the measure of the angle of rotation if vertex 2 is moved counterclockwise to the current position of vertex 6?
20. If vertex 5 is rotated 280° counterclockwise, find its new position.



9-4 Tessellations

See pages 483–488.

Concept Summary

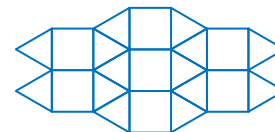
- A tessellation is a repetitious pattern that covers a plane without overlap.
- A regular tessellation contains the same combination of shapes and angles at every vertex.

Example

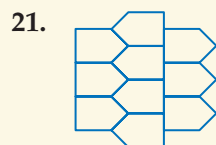
Classify the tessellation at the right.

The tessellation is uniform, because at each vertex there are two squares and three equilateral triangles. Both the square and equilateral triangle are regular polygons.

Since there is more than one regular polygon in the tessellation, it is a semi-regular tessellation.



Exercises Determine whether each pattern is a tessellation. If so, describe it as *uniform*, *not uniform*, *regular*, or *semi-regular*. See Example 3 on page 485.



Determine whether each regular polygon will tessellate the plane. Explain.

See Example 1 on page 484.

24. pentagon
25. triangle
26. decagon

9-5 Dilations

See pages
490–497.

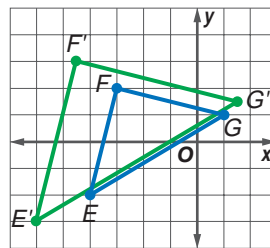
Concept Summary

- Dilations can be enlargements, reductions, or congruence transformations.

Example

Triangle EFG has vertices $E(-4, -2)$, $F(-3, 2)$, and $G(1, 1)$. Find the image of $\triangle EFG$ after a dilation centered at the origin with a scale factor of $\frac{3}{2}$.

Preimage (x, y)	Image $(\frac{3}{2}x, \frac{3}{2}y)$
$E(-4, -2)$	$E'(-6, -3)$
$F(-3, 2)$	$F'(-\frac{9}{2}, 3)$
$G(1, 1)$	$G'(\frac{3}{2}, \frac{3}{2})$



Exercises Find the measure of the dilation image $\overline{C'D'}$ or preimage of \overline{CD} using the given scale factor. See Example 1 on page 491.

27. $CD = 8, r = 3$ 28. $CD = \frac{2}{3}, r = -6$ 29. $C'D' = 24, r = 6$
 30. $C'D' = 60, r = \frac{10}{3}$ 31. $CD = 12, r = -\frac{5}{6}$ 32. $C'D' = \frac{55}{2}, r = \frac{5}{4}$

Find the image of each polygon, given the vertices, after a dilation centered at the origin with a scale factor of -2 . See Example 3 on page 492.

33. $P(-1, 3), Q(2, 2), R(1, -1)$ 34. $E(-3, 2), F(1, 2), G(1, -2), H(-3, -2)$

9-6 Vectors

See pages
498–505.

Concept Summary

- A vector is a quantity with both magnitude and direction.
- Vectors can be used to translate figures on the coordinate plane.

Example

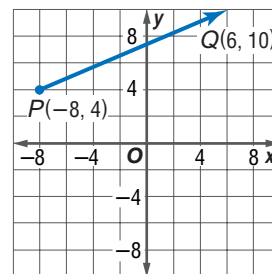
Find the magnitude and direction of \overrightarrow{PQ} for $P(-8, 4)$ and $Q(6, 10)$.

Find the magnitude.

$$\begin{aligned} |\overrightarrow{PQ}| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\ &= \sqrt{(6 + 8)^2 + (10 - 4)^2} && x_1 = -8, y_1 = 4, x_2 = 6, y_2 = 10 \\ &= \sqrt{232} && \text{Simplify.} \\ &\approx 15.2 && \text{Use a calculator.} \end{aligned}$$

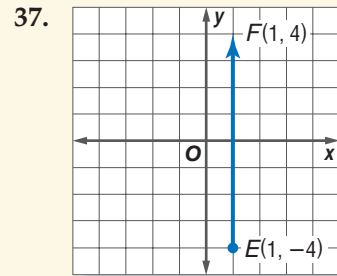
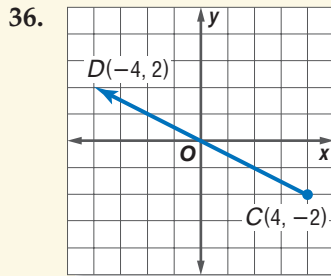
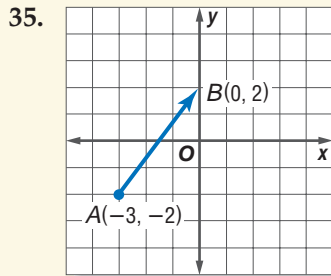
Find the direction.

$$\begin{aligned} \tan P &= \frac{y_2 - y_1}{x_2 - x_1} && \frac{\text{length of opposite side}}{\text{length of adjacent side}} \\ &= \frac{10 - 4}{6 + 8} && \text{Substitution} \\ &= \frac{6}{14} \text{ or } \frac{3}{7} && \text{Simplify.} \end{aligned} \quad \left| \quad \begin{aligned} m\angle P &= \tan^{-1} \frac{3}{7} \\ &\approx 23.2 && \text{Use a calculator.} \end{aligned}$$



- Extra Practice, see pages 771–773.
- Mixed Problem Solving, see page 790.

Exercises Write the component form of each vector. See Example 1 on page 498.



Find the magnitude and direction of \overrightarrow{AB} for the given coordinates.

See Example 2 on page 499.

38. $A(-6, 4), B(-9, -3)$

39. $A(8, 5), B(-5, -2)$

40. $A(-14, 2), B(15, -5)$

41. $A(16, 40), B(-45, 0)$

9-7 Transformations with Matrices

See pages
506–511.

Concept Summary

- The vertices of a polygon can be represented by a vertex matrix.
- Matrix operations can be used to perform transformations.

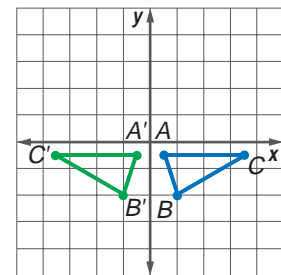
Example

Use a matrix to find the coordinates of the vertices of the image of $\triangle ABC$ with $A(1, -1), B(2, -4), C(7, -1)$ after a reflection in the y -axis.

Write the ordered pairs in a vertex matrix. Then use a calculator to multiply the vertex matrix by the reflection matrix.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 7 \\ -1 & -4 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -7 \\ -1 & -4 & -1 \end{bmatrix}$$

The coordinates of $\triangle A'B'C'$ are $A'(-1, -1), B'(-2, -4)$, and $C'(-7, -1)$.



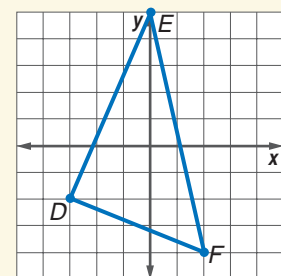
Exercises Find the coordinates of the image after the stated transformation. See Example 1 on page 506.

42. translation $(x, y) \rightarrow (x - 3, y - 6)$

43. dilation by scale factor $\frac{4}{5}$

44. reflection in the line $y = x$

45. rotation 270° counterclockwise about the origin



Find the coordinates of the image after the stated transformations. See Examples 2–4 on pages 507 and 508.

46. $\triangle PQR$ with $P(9, 2), Q(1, -1)$, and $R(4, 5)$; $(x, y) \rightarrow (x + 2, y - 5)$, then a reflection in the x -axis

47. $\square WXYZ$ with $W(-8, 1), X(-2, 3), Y(-1, 0)$, and $Z(-6, -3)$; a rotation 180° counterclockwise, then a dilation by scale factor -2

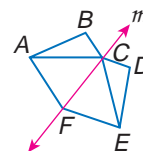
Vocabulary and Concepts

Choose the correct term to complete each sentence.

- If a dilation does not change the size of the object, then it is a(n) (*isometry*, *reflection*).
- Tessellations with the same shapes and angles at each vertex are called (*uniform*, *regular*).
- A vector multiplied by a (*vector*, *scalar*) results in another vector.

Skills and Applications

Name the reflected image of each figure under a reflection in line m .

4. A 5. \overline{BC} 6. $\triangle DCE$ 

COORDINATE GEOMETRY Graph each figure and its image under the given translation.

- $\triangle PQR$ with $P(-3, 5)$, $Q(-2, 1)$, and $R(-4, 2)$ under the translation right 3 units and up 1 unit
- Parallelogram $WXYZ$ with $W(-2, -5)$, $X(1, -5)$, $Y(2, -2)$, and $Z(-1, -2)$ under the translation up 5 units and left 3 units
- \overline{FG} with $F(3, 5)$ and $G(6, -1)$ under the translation $(x, y) \rightarrow (x - 4, y - 1)$

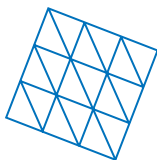
Draw the rotation image of each triangle by reflecting the triangles in the given lines.

State the coordinates of the rotation image and the angle of rotation.

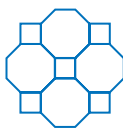
- $\triangle JKL$ with $J(-1, -2)$, $K(-3, -4)$, $L(1, -4)$ reflected in the y -axis and then the x -axis
- $\triangle ABC$ with $A(-3, -2)$, $B(-1, 1)$, $C(3, -1)$ reflected in the line $y = x$ and then the line $y = -x$
- $\triangle RST$ with $R(1, 6)$, $S(1, 1)$, $T(3, -2)$ reflected in the y -axis and then the line $y = x$

Determine whether each pattern is a tessellation. If so, describe it as *uniform*, *not uniform*, *regular*, or *semi-regular*.

13.



14.



15.



Find the measure of the dilation image $\overline{M'N'}$ or preimage of \overline{MN} using the given scale factor.

16. $MN = 5, r = 4$

17. $MN = 8, r = \frac{1}{4}$

18. $M'N' = 36, r = 3$

19. $MN = 9, r = -\frac{1}{5}$

20. $M'N' = 20, r = \frac{2}{3}$

21. $M'N' = \frac{29}{5}, r = -\frac{3}{5}$

Find the magnitude and direction of each vector.

22. $\vec{v} = \langle -3, 2 \rangle$

23. $\vec{w} = \langle -6, -8 \rangle$

- TRAVEL** In trying to calculate how far she must travel for an appointment, Gunja measured the distance between Richmond, Virginia, and Charlotte, North Carolina, on a map. The distance on the map was 2.25 inches, and the scale factor was 1 inch equals 150 miles. How far must she travel?

- STANDARDIZED TEST PRACTICE** What reflections could be used to create the image $(3, 4)$ from $(3, -4)$?

I. reflection in the x -axisII. reflection in the y -axis

III. reflection in the origin

(A) I only

(B) III only

(C) I and III

(D) I and II



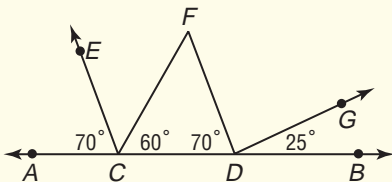
Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. Ms. Lee told her students, "If you do not get enough rest, you will be tired. If you are tired, you will not be able to concentrate." Which of the following is a logical conclusion that could follow Ms. Lee's statements? (Lesson 2-4)

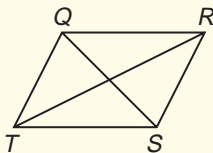
- (A) If you get enough rest, you will be tired.
- (B) If you are tired, you will be able to concentrate.
- (C) If you do not get enough rest, you will be able to concentrate.
- (D) If you do not get enough rest, you will not be able to concentrate.

2. Which of the following statements is true? (Lesson 3-5)



- (A) $\overline{CE} \parallel \overline{DF}$
- (B) $\overline{CF} \parallel \overline{DG}$
- (C) $\overline{CF} \cong \overline{DF}$
- (D) $\overline{CE} \cong \overline{DF}$

3. Which of the following would *not* prove that quadrilateral $QRST$ is a parallelogram? (Lesson 8-2)

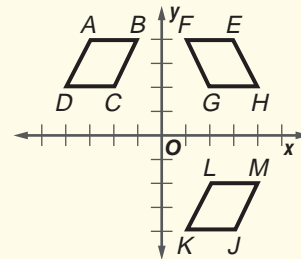


- (A) Both pairs of opposite angles are congruent.
- (B) Both pairs of opposite sides are parallel.
- (C) Diagonals bisect each other.
- (D) A pair of opposite sides is congruent.

4. If $Q(4, 2)$ is reflected in the y -axis, what will be the coordinates of Q' ? (Lesson 9-1)

- (A) $(-4, -2)$
- (B) $(-4, 2)$
- (C) $(2, -4)$
- (D) $(2, 4)$

5. Which of the following statements about the figures below is true? (Lesson 9-2)



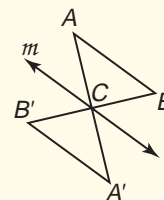
- (A) Parallelogram $JKLM$ is a reflection image of $\square ABCD$.
- (B) Parallelogram $EFGH$ is a translation image of $\square ABCD$.
- (C) Parallelogram $JKLM$ is a translation image of $\square EFGH$.
- (D) Parallelogram $JKLM$ is a translation image of $\square ABCD$.

6. Which of the following is **not** necessarily preserved in a congruence transformation? (Lesson 9-2)

- (A) angle and distance measure
- (B) orientation
- (C) collinearity
- (D) betweenness of points

7. Which transformation is used to map $\triangle ABC$ to $\triangle A'B'C$ with respect to line m ? (Lesson 9-4)

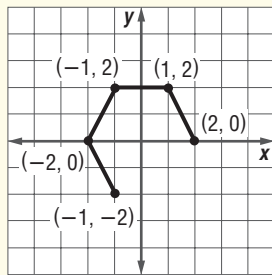
- (A) rotation
- (B) reflection
- (C) dilation
- (D) translation



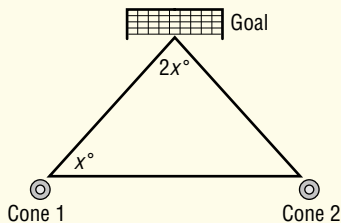
Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

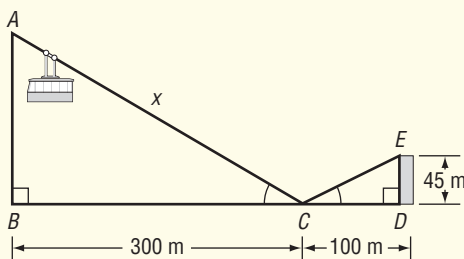
8. A new logo was designed for GEO Company. The logo is shaped like a symmetrical hexagon. What are the coordinates of the missing vertex of the logo? (Lesson 1-1)



9. A soccer coach is having her players practice penalty kicks. She places two cones equidistant from the goal and asks the players to line up behind each cone. What is the value of x ? (Lesson 4-6)



10. A steel cable, which supports a tram, needs to be replaced. To determine the length x of the cable currently in use, the engineer makes several measurements and draws the diagram below of two right triangles, $\triangle ABC$ and $\triangle EDC$. If $m\angle ACB = m\angle ECD$, what is the length x of the cable currently in use? Round the result to the nearest meter. (Lesson 6-3)



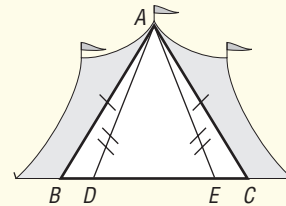
Test-Taking Tip

Question 4 Always check your work for careless errors. To check your answer to this question, remember the following rule. In a reflection over the x -axis, the x -coordinate remains the same, and the y -coordinate changes its sign. In a reflection over the y -axis, the y -coordinate remains the same, and the x -coordinate changes its sign.

Part 3 Open Ended

Record your answers on a sheet of paper. Show your work.

11. Kelli drew the diagram below to show the front view of a circus tent. Prove that $\triangle ABD$ is congruent to $\triangle ACE$. (Lessons 4-5 and 4-6)



12. Paul is studying to become a landscape architect. He drew a map view of a park on the following vertices: $Q(2, 2)$, $R(-2, 4)$, $S(-3, -2)$, and $T(3, -4)$
- On a coordinate plane, graph quadrilateral $QRST$. (Prerequisite Skill)
 - Paul's original drawing appears small on his paper. His instructor says that he should dilate the image with the origin as center and a scale factor of 2. Graph and label the coordinates of the dilation image $Q'R'S'T'$. (Lesson 9-5)
 - Explain how Paul can determine the coordinates of the vertices of $Q'R'S'T'$ without using a coordinate plane. Use one of the vertices for a demonstration of your method. (Lesson 9-5)
 - Dilations are similarity transformations. What properties are preserved during an enlargement? reduction? congruence transformation? (Lesson 9-5)